College Board Standards for College Success™

Mathematics and Statistics

Adapted for Integrated Curricula

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Introduction to Standards for College Success

The College Board has developed standards for mathematics and statistics to help states, school districts, and schools provide all students with the rigorous education that will prepare them for success in college, opportunity in the workplace, and effective participation in civic life. The College Board's commitment to this project is founded on the belief that all students can meet high expectations for academic performance when they are taught to high standards by qualified teachers.

College Board programs and services have supported the transition from high school to college for more than 100 years. Advanced Placement Program® (AP®) courses enable students to transition into college-level study when they are ready, even while still in high school. The SAT® Reasoning Test™, the SAT Subject Tests™, and the PSAT/NMSQT® all measure content knowledge and critical thinking and reasoning skills that are foundations for success in college. The College Board Standards for College Success makes explicit these college readiness skills so that states, school districts, and schools can better align their educational programs to clear definitions of college readiness.

Preparing students for college *before* they graduate from high school is critical to students' completing a college degree. Most college students who take remedial courses fail to earn a bachelor's degree (Adelman, 2004). To reduce the need for remediation in college, K–12 educational systems need clear and specific definitions of the knowledge and skills that students should develop by the time they graduate in order to be prepared for college success. By aligning curriculum, instruction, assessment, and professional development to clear definitions of college readiness, schools can help reduce the need for remediation in college and close achievement gaps among student groups, ultimately increasing the likelihood that students will complete a college degree.

The design of the College Board Standards for College Success reflects the specific purposes of this framework—to vertically align curriculum, instruction, assessment, and professional development across six levels beginning in middle school leading to AP and college readiness. The College Board Standards for College Success is, therefore, more specific than most standards documents because it is intended to provide sufficient guidance for curriculum supervisors and teachers to design instruction and assessments in middle school and high school that lead toward AP and college readiness. The College Board uses these frameworks to align its own curriculum and assessment programs, including SpringBoard™, to college readiness. States and districts interested in integrating SpringBoard and AP into a program of college readiness preparation can use the College Board Standards for College Success as a guiding framework.

Development of the Mathematics and Statistics College Board Standards for College Success

The College Board initiated the effort to develop the Mathematics and Statistics College Board Standards for College Success in 2003. To guide the process, the College Board convened the Mathematics and Statistics Standards Advisory Committee, comprising middle school and high school teachers, college faculty, subject matter experts, assessment specialists, teacher education faculty, and curriculum experts with experience developing content standards for states and national professional organizations.

The Standards Advisory Committee was charged with developing standards that express a progression of student performances leading to well-defined criteria that characterize college readiness. All students need and deserve an academically rigorous high school program of study to succeed, whether they choose to attend college or seek a well-paying entry-level job with opportunity for advancement in today's knowledge-based global economy. Recent analyses have shown that the knowledge and skills required for college success are comparable to the knowledge and skills required by entry-level jobs with opportunity for advancement (Carnevale & Desrochers, 2004; Friedman, 2005; Meeder & Solares, 2006; ACT, 2006). Today, all students need advanced mathematics knowledge and skills to succeed, whatever their chosen life path. Moreover, the increased role of technology and science in our everyday lives means that students graduating from high school need a deeper understanding of the collection, analysis, and interpretation of data in order to participate effectively in civic life.

The Standards Advisory Committee first defined standards for college readiness in mathematics and statistics by reviewing the assessment frameworks for relevant AP examinations, the SAT, the PSAT/NMSQT, the College-Level Examination Program® (CLEP®) examinations, and selected university placement programs. The Committee also reviewed the results of several surveys and course content analyses conducted by the College Board to provide empirical validation of the emerging definitions of college readiness.

In mathematics and statistics, the College Board surveyed 924 college mathematics faculty and 1,545 high school mathematics teachers to determine the knowledge and skills they believed were critical to success in first-year college mathematics courses. In addition to the mathematics surveys, College Board researchers analyzed course syllabi, assignments, and student work samples from each institution represented in the survey, conducted eight comprehensive case studies of first-year college mathematics courses, and conducted focus groups to further validate and contextualize the findings (Conley, Aspengren, Gallagher, & Nies, 2006; King, Huff, Michaelides, & Delgado, 2006). The College Board also surveyed 96 college faculty and 63 high school teachers who teach precalculus to determine the knowledge and skills that should be developed in a precalculus course to lay the foundations for success in calculus (College Board, 2005).

Further empirical data were developed through a three-year national study sponsored by the Association of American Universities (AAU) and conducted by the Center for Educational Policy Research (CEPR) at the University of Oregon. This study surveyed more than 400 college faculty at nine AAU universities throughout the nation to define the knowledge and skills necessary for successful performance in entry-level college courses.

Definitions of college readiness gathered through these surveys, course analyses, and case studies represent the most rigorously researched, empirically validated definitions of college readiness available.

Having established clear and specific definitions of the knowledge and skills that students need to succeed in college, the Standards Advisory Committee articulated a developmental progression of student performance expectations that would lead students to being prepared for college-level work. These performance expectations were first articulated within a sequence of six courses in the well-known sequence of Middle School Math I and II, Algebra I, Geometry, Algebra II, and Precalculus. Articulating performance expectations for these six college-preparatory courses in mathematics entailed reviewing existing curricula as defined by selected state content standards, selected district curriculum frameworks, textbooks, and assessment frameworks for selected state examinations, the National Assessment of Educational Progress (NAEP), and the Trends in International Mathematics and Science Study (TIMSS). The Committee sought to align the Mathematics and Statistics College Board Standards for College Success to these curriculum and assessment frameworks while also ensuring that the developmental progression of learning objectives outlined in the Standards would lead to the targeted college readiness expectations.

Integral to this process was reviewing other national content standards and guidelines in mathematics and statistics. The Standards Advisory Committee reviewed the *Principles and Standards for School Mathematics* published by the National Council of Teachers of Mathematics (2000); the *PreK–12 Guidelines for Assessment and Instruction in Statistics Education* developed by the American Statistical Association (2005); the American Diploma Project *Benchmarks* published by Achieve, Inc. (2004); and the *Knowledge and Skills for University Success* published by Standards for Success (2003).

As part of this process, the Mathematics and Statistics Standards were compared with the state frameworks and curricular materials associated with several of the extant middle and secondary school programs loosely based on the NCTM Standards and whose development, in some cases, was supported by the National Science Foundation. These curricula, sometimes referred to as integrated curricula, place a greater emphasis on applications of mathematics than more traditionally oriented curricula and present content in a different sequential order in many cases.

As a result of these comparisons, the content of the Mathematics and Statistics College Board Standards for College Success was adapted for integrated curricula by resequencing the content in the standards document through additional meetings of members of the Standards Advisory Committee. As such, the present document reflects the same content goals, but illustrates the manner in which this content might be rearranged if taught in an integrated program, rather than in the traditional course-based curriculum.

The College Board would like to acknowledge the following national professional organizations that assisted in providing individual or organizational committee reviews of drafts of the Mathematics and Statistics College Board Standards for College Success. These organizations represent key constituencies committed to improving K–12 and postsecondary teaching and learning, and the College Board is grateful to have received input reflecting each organization's perspective, experience, and expertise. Their wide-ranging comments allowed the Mathematics and Statistics Standards Advisory Committee an opportunity to sharpen the document, to revisit issues on which diverse opinions were voiced, and to provide additional commentary explaining the final version of the Standards for College Success. The members of the Standards Advisory Committee appreciated and valued the comments of all the organizations and individuals who provided input.

A listing of individuals involved in this process is provided below. However, the College Board is solely responsible for the final version of the Mathematics and Statistics College Board Standards for College Success, and the reviews provided by these organizations do not represent endorsement of or agreement with the College Board Standards.

- National Council of Teachers of Mathematics (NCTM)
- Mathematical Association of America (MAA)
- American Mathematical Society (AMS)
- American Statistical Association (ASA)
- American Mathematical Association of Two-Year Colleges (AMATYC)
- Association of State Supervisors of Mathematics (ASSM)
- Achieve, Inc.

College Board Mathematics and Statistics Standards Advisory Committee

Members of the College Board Mathematics and Statistics Standards Advisory Committee convened for more than a dozen working meetings throughout the course of this project and worked hundreds of additional hours to draft, review, and revise the Standards for College Success. The College Board is grateful for their commitment and dedication to this effort.

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Introduction to Mathematics and Statistics

Mathematics and statistics standards documents (National Council of Teachers of Mathematics, 2000; American Statistical Association, 2005) describe the teaching and learning of mathematics and statistics as an integrated collection of processes and content elements. While mathematics and statistics are separate disciplines under the umbrella of the broad mathematical sciences, they share a number of concepts that can serve as focal points in organizing instruction. Mathematical thinking and statistical thinking both involve logical reasoning and recognizing patterns, but mathematical thinking is more often deterministic and statistical thinking is usually probabilistic in nature. Students need to learn both ways of thinking. Mathematical reasoning involves abstraction and generalization. Statistical reasoning involves data in a context, variation, and chance. The sequencing of this knowledge in these standards, within courses, provides a solid foundation for the middle school and high school curriculum in mathematics and statistics.

The Mathematics and Statistics College Board Standards for College Success is based on a firm belief that excessive time spent reviewing topics year after year while only minimally introducing new topics is not conducive to effective mathematics instruction. Further, students profit from learning in settings developed around rich problem contexts that engage students in exploring, problem solving, generalizing mathematical concepts and algorithms, and developing their capabilities to *do mathematics*.

Quality programs that adequately meet the needs of students as they work to master the content outlined in this document require that students and teachers have adequate instructional time allotted to mathematics within the school day. The College Board strongly recommends a commitment of the equivalent of 55 minutes or more of instructional time for mathematics every day, especially at the middle school level. Although this recommendation goes beyond what is currently allotted for mathematics in many schools, this amount of time for mathematics is critical if students are to be successful in the mathematics and statistics necessary for college success. The allotment of less time per day creates conditions where middle school students, in particular, are unlikely to develop the mathematical understanding, problem-solving abilities, reasoning, or knowledge base to support learning in higher-level mathematics courses. How the allocated time is used is as important as the number of minutes allotted. Mathematics instructional time needs to be focused on engaging students in learning challenging mathematics, developing mathematical understanding, and becoming flexible problem solvers.

In an integrated curriculum, the content from the beginning of middle school through the end of high school involves strands that are generally similar to those found in the more traditional organization of curriculum, but the sequencing and the grade levels at which students develop proficiency with particular concepts or algorithms may differ.

Teachers of integrated curricula, as well as of traditionally organized curricula, need a clear view of how their instruction fits within the curriculum—including what students are expected to know as prerequisite knowledge and at critical points within the study of mathematics and statistics. With this as a base, teachers can provide students with varied opportunities to learn mathematics and to work on wide-ranging problems that allow them to build skill in applying their mathematical knowledge and problem-solving strategies to situations in mathematics and the real world.

Central to the knowledge and skills developed in the middle school and high school years are the following broad classes of concepts, procedures, and processes:

- Operations and Equivalent Representations
- Algebraic Manipulation Skills
- Quantity and Measurement
- Proportionality
- Relations, Patterns, and Functions
- Shape and Transformation
- Data and Variation
- Chance, Fairness, and Risk

Students' study of mathematics in middle school begins with a basic understanding of fractions and the operations defined on them. The first two years of the middle school curriculum focus on developing a deep understanding of the rational numbers, their different representations, and the connections between these numbers and other number systems and operations students have studied. Students entering the middle grades are expected to have computational fluency with whole-number operations and to be developing a broad understanding of addition, subtraction, and multiplication with rational numbers. Students may still lack an understanding of and computational fluency with multiplication and division of rational numbers. Developing that understanding and fluency will require additional learning experiences and discussion.

Paralleling this study is the development of proportionality. Students examine ratios and rates, and see that these describe particular, and then general, forms of comparison. As with the study of mathematics more generally, the study of proportionality should go beyond learning a set of procedures for solving basic ratio and percent problems. It involves developing conceptual understanding that supports flexible application. Proportionality serves as a critical foundation for algebra and the rest of the high school mathematics curriculum. As with rational numbers, in ratios and proportional relationships the question of representation is key, especially the concept of equivalence and its uses.

This content supports the study of similarity, and its special case of congruence, of geometric figures. The algebra strand of the curriculum works to help students move from examining patterns of change and growth to writing symbolic expressions modeling those patterns. From here, students progress to developing the concept of a function and understanding the relationship between the variables involved. As students become conversant with the use of variables and expressions, they extend their understanding of the properties of operations, equations, and inequalities. As their conception of algebraic relationships grows, students also are expected to develop fluency in the manipulation of algebraic equations, formulas, and function notation to support their representing and solving of problems where this knowledge applies.

Experiences with data investigations support the focus on the identification and exploration of questions that can be addressed through the collection and analysis of data. Students' capabilities to analyze data grow as their mathematical maturity increases. Students learn to collect, analyze, and interpret data describing one characteristic measured on each subject and, later, examine situations calling for bivariate data. Investigations with real data, collected by students, give them opportunities to examine the distributions of individual variables, and to explore the association between two variables in a variety of situations drawn from a broad range of disciplines. Analyzing one variable at a time, students grow in their ability to select and use summary numbers such as the mean, median, range, percentiles, interquartile range, and standard deviation to describe the distribution of the data and in their ability to make comparisons between two distributions using both graphs and summary numbers. Analyzing two variables at a time, students learn to discern associations through numerical patterns and visually through scatterplots. The curriculum connects the understanding of probability with the work done on proportions, by representing probabilities numerically and estimating them from data and from models. Students determine the sample space for one-stage and for two-independent-stage experiments.

As students transition to Integrated Mathematics III and IV, which contain the content normally included in courses such as Algebra I and Geometry, the focus of the curriculum shifts to the study of linear and nonlinear equations and systems of linear equations and the application of such relationships in geometric settings. The study of geometry also focuses on the role of transformations and the development of the measurement formulas for circumference, area, surface area, and volume. This formalizes students' previous work with the measurement formulas in earlier grades. Across the high school curriculum, in settings within both algebra and geometry, emphasis is placed on enhancing students' reasoning skills with a major focus on justification and proof. Integrated Mathematics V and VI contain much of the content normally studied in Algebra II and Precalculus. This content extends the study of polynomial, rational, and radical expressions and equations, and introduces a variety of functions: exponential, logarithmic, and trigonometric. These functions, their applications, and the associated operations and solution methods are developed and employed in a wide range of settings.

In the areas of data analysis and probability, high school students learn about principles of study design, sampling methods, and types of statistical studies, including surveys, observational studies, and experiments. Students examine the role of random selection in assuring that samples from a population of interest appropriately represent that population, and they learn the importance of random allocation in experimental studies for reducing bias in the comparison of treatments. In their study of probability, students develop organized methods for representing one- and two-stage experiments, and for delineating the possible outcomes and events associated with such experiments. They recognize mutually exclusive and independent events, distinguishing among them in assigning probabilities. They also understand and use conditional probabilities. They can use the multiplication rule (Fundamental Property of Counting), permutations, and combinations to find specific counts. Students also learn to solve problems by using the normal and the binomial distributions to estimate probabilities.

In their study of mathematical structures, students again expand their view of functions: from that of expressions representing patterns to a more-general picture of functions as mathematical objects with associated operations and interpretations. Students gradually see functions as special classes of mathematical objects that represent specific relationships. They can distinguish among functions and other relationships between variables, such as those geometrically represented by ellipses and hyperbolas. They recognize the need to restrict the domain of inverse trigonometric relations to guarantee the existence of a relationship that is a function.

Traditionally, this material has been organized in a fashion that separates algebraic and geometric content into distinct courses, while ignoring the content from important emerging areas, such as data analysis and probability. Integrated approaches modify a traditional sequence of instruction by breaking down artificial walls between algebra and geometry and teaching the varied curricular strands concurrently if they are mutually supportive or linked by underlying concepts. As such, the following presentation of the Mathematics and Statistics College Board Standards for College Success Adapted for Integrated Curricula provides a rearrangement of the content found in the Standards for College Success to support an integrated curricular approach.

The Roles of Content and Processes in Mathematics and Statistics Instruction

The Mathematics and Statistics College Board Standards for College Success Adapted for Integrated Curricula describes mathematics and statistics proficiencies in ways that combine process expectations with content knowledge. The resulting emphasis on content knowledge provides teachers, curriculum directors, assessment developers, and others involved with the design and teaching of school mathematics and statistics at the classroom level with a detailed outline of content standards and objectives by course level. This specificity enables teachers and administrators to carefully plan and assess students' progress against a set of standards tied to courses.

While the Mathematics and Statistics College Board Standards for College Success Adapted for Integrated Curricula does not explicitly contain standards addressing the cognitive processes of problem solving; reasoning and proof; making and translating among representations; developing connections within and among mathematics, statistics, and their applications; and developing and using a variety of communication skills in learning about and explaining mathematical contexts, the Standards links the content to these processes in the objectives and performance expectations. The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (2000) provides a thorough elaboration of these processes and their interaction with the learning of content across the mathematics and statistics curriculum.

The Mathematics and Statistics College Board Standards for College Success Adapted for Integrated Curricula includes a greater focus on statistics and probability than is currently seen in most classrooms. This decision was based on the ever-growing influence of statistics and probability in everyday life and in scientific and technological applications across our society. It is fundamental that students understand the basic concepts and applications of statistics and probability and appreciate the modes of reasoning that are used in these fields. The College Board recognizes that the inclusion of topics from probability and statistics will vie for precious time in the overall mathematical sciences curriculum; however, the College Board believes that this information can be as important to students as any mathematical topic that might be displaced. The present and emerging uses of statistics and probability in our society have made these fields a part of the "new basics" for all students. Such knowledge is critical for students' success in quantitatively based college courses, as well as effective participation in civic life.

The Standards also links geometry and measurement into one content domain. This decision was made because of the difficulty in separating work dealing with geometric figures and regions and that dealing with their associated measures. Because these topics are combined within the secondary curriculum, standards for them should also be combined. Both of these changes from tradition are consistent with the National Assessment Governing Board's *Mathematics Framework* (2004) for evaluating the mathematical knowledge of the nation's youth in grades 4, 8, and 12 on the National Assessment of Educational Progress (NAEP). The table below details the proportion of assessment items included in the NAEP program for students at each of the target grade levels. In many states, these suggested weightings of content areas are extended and elaborated at grade levels in between those shown.

<table>
<tr><th>Content Area</th><th>Grade 4
(% of items)</th><th>Grade 8
(% of items)</th><th>Grade 12
(% of items)</th></tr>
<tr><td>Number Properties and Operations</td><td>40</td><td>20</td><td>10</td></tr>
<tr><td>Measurement</td><td>20</td><td>15</td><td rowspan="2">30</td></tr>
<tr><td>Geometry</td><td>15</td><td>20</td></tr>
<tr><td>Algebra</td><td>15</td><td>30</td><td>35</td></tr>
<tr><td>Data Analysis and Probability</td><td>10</td><td>15</td><td>25</td></tr>
</table>

While the number of standards, objectives, and performance expectations in this document do not directly reflect the percentages shown for the National Assessment, the Standards, as a whole, provides a guide for a school program that reflects the role played by branches of mathematics and statistics and for what might be an appropriate balance among these areas within a contemporary curriculum.

To further support teachers as they help students develop a knowledge base in the various areas of practice, a glossary is provided at the end of the document, as researchers, teachers, and other experts sometimes define terms differently. Terms that appear in the glossary are underlined once in each course in which they appear, at their first significant occurrence following the course introduction.

Helping students develop their knowledge of and proficiency with mathematics and statistics includes discerning and affirming the knowledge and skills that each student brings to the classroom. To enable all students to succeed in mathematics and statistics classrooms, it is essential that we recognize and affirm the knowledge and practices that students bring with them, and set realistic, but challenging, expectations for mathematics learning that apply to students and teachers across multiple language, cultural, and ethnic groups. Students must learn to represent, discuss, and question mathematical content along with working to progressively develop their knowledge of and ability to successfully apply mathematics. Schools and teachers who recognize and build on students' unique backgrounds, knowledge, and skills will be better able to help their students acquire these expected competencies in mathematics and statistics—critical requirements for success in college and the workplace.

Using the Mathematics and Statistics College Board Standards for College Success Adapted for Integrated Curricula to Design Curriculum and Instruction

Unlike most state standards for mathematics, which historically have not been developed explicitly to prepare students for college admission and success, the Mathematics and Statistics College Board Standards for College Success Adapted for Integrated Curricula sets culminating expectations for student proficiency that are anchored in definitions of college readiness developed through analyses of first-year college courses, surveys of college faculty, and analyses of college admissions and placement examinations. Expectations for each of the courses building toward these culminating expectations were also developed by reviewing national standards documents, state standards frameworks, selected district curriculum frameworks, and textbooks. The Standards for College Success thus articulates a continuum of student performance expectations that both align to developmentally appropriate benchmarks of student proficiency in middle school and high school and prepare students for college success.

This version of the College Board Standards for College Success defines expectations in terms of a sequence of courses following an integrated curricular design. As such, the expectations speak to what students in the large should be expected to know and be able to do at particular points in the curriculum. The College Board is aware that individual students progress at different rates and that these rates may vary for an individual depending on the actual subject matter being examined at a given time. As a result, schools and individual teachers will make deviations from the standards, objectives, and performance expectations listed based on their analysis of local constraints and student needs. However, any major shortfall from the content listed in the Standards will result in a program that fails to adequately prepare students for a seamless transition into the workforce or collegiate programs having quantitative prerequisites. Preparation for such transitions requires that students have developed a solid understanding of the mathematics and statistics concepts, principles, and skills outlined in these Standards and have had opportunities to apply this knowledge in a rich variety of contexts.

Throughout this document, the verb "develop" is used as a descriptor of performance expectations for students' study of mathematics. This word is deliberately selected to indicate that the student sees the concept behind the content carefully developed through teacher presentations and student investigations. But it further indicates that the student "develops" a deep understanding of the content focused upon—the student recognizes when a concept applies, can identify examples and nonexamples of it, can correctly and efficiently perform the related algorithmic procedures, sketches or constructs physical representations to investigate instances of the concept, and can use the concept appropriately in modeling mathematical and real-world applications. While this document does not focus directly on the instructional acts related to the content in the classroom, the reader must infer from the verbs what might constitute quality instruction related to these content goals.

Because the course levels associated with the course titles provide a recognized structure within many schools, teachers may locate their students among the courses and differentiate instruction to support and challenge students in ways that are most productive for each student's individual growth. The following sections present the standards for six integrated course in mathematics beginning with grade 6. These six courses outline the mathematics and statistics content students should encounter to prepare them to be successful in the workplace or in the study of collegiate mathematics without the need for remediation. Students completing these six courses by the end of the eleventh grade are well prepared for the study of Advanced Placement Calculus in the twelfth grade. Students following this sequence of courses may take Advanced Placement Statistics following Integrated Mathematics V or at any later time in their studies. Individual districts and schools may choose to develop programs focusing on this content that cover six or seven years of study, as individual curricula need to be determined based on students' mathematical knowledge at the point of entry into the middle school and their readiness to understand and benefit from the recommended content.

Middle Grades Mathematics

Integrated Mathematics I and II define content often found in grades 6 and 7, and together, these courses provide a transition from elementary school mathematics to the integration of algebra and geometry topics studied in grade 8. As such, they provide a bridge from the study of mathematics centered on numbers and operations to a study of secondary school mathematics centered on algebra and functions. This is an essential transition for students, as it provides the expansion of number and operation concepts from whole numbers to real numbers; geometry relationships from two dimensions to space; data from questions focusing on a single data value for each sampled unit to situations where one is comparing two data values per subject or two groups on a single variable; and algebra from arithmetic and geometric patterns to variables, expressions, and equations. Supporting a great deal of this work is the growth of proportional reasoning—the reasoning of multiplicative structures. This transition also spans the growth from concrete reasoning to semi-abstract reasoning.

The first two mathematics courses in the College Board's Standards for College Success are the same for both traditionally oriented and integrated approaches, as the curriculum at these levels has followed an integrated approach for the past decade.

Integrated Mathematics I and II encapsulate three years of pre–Algebra I mathematics traditionally studied in grades 6–8 into two courses, and these courses outline content that provides a basis for the study of algebra in grade 8. This reorganization does not suggest that portions of an existing three-year program should be covered before the middle grades. Rather, it revises the traditional middle school mathematics courses to eliminate the excessive repetition that often characterizes mathematics curricula at these levels (McKnight et al., 1987; Schmidt et al., 2001). This elimination of redundancy allows an expanded focus on data analysis and probability. The recommended content for the two courses preserves the core of the three years of content traditionally taught in the middle school grades. Regardless of the curricular design, the mathematics covered in these courses is important, and students must develop proficiency in this content before beginning Integrated Mathematics III or Algebra I.

Districts and schools using these standards to guide curricular design may benefit from reviewing other standards frameworks that offer guidance on sequencing instruction for mathematics and statistics in the middle grades, including the National Council of Teachers of Mathematics' *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence* (2006) and the American Statistical Association's *Guidelines for Assessment and Instruction in Statistics Education* (2005).

Completion of Integrated Mathematics I and II in grades 6 and 7 positions students preparing for mathematics-dependent courses at the collegiate level with the opportunity take AP Statistics following Integrated Mathematics V and AP Calculus following the completion of Integrated Mathematics VI. Other students may benefit from covering the material found in Integrated

Mathematics I and II over a three-year span and completing Integrated Mathematics III through Integrated Mathematics VI (or Algebra I through Precalculus) in grades 9–12. These students also will be well prepared for success in college. All students, regardless of the sequence of mathematics courses they take in high school, must be committed to studying mathematics each year in grades 9 through 12.

In addition to the content listed in the following curricular outlines for integrated middle school mathematics, students must develop a broad range of skills in reasoning and proof, problem solving, and representing mathematical concepts and principles, as well as learn to connect concepts in mathematics within and outside mathematics itself. Furthermore, students need to develop the ability to communicate (read, write, and speak) about mathematics and its applications in a diverse set of contexts as mathematics is studied and applied. Students should also begin to incorporate technology into their reasoning and problem solving, making use of graphing calculators, spreadsheets, dynamic geometry software, and statistical software in investigating mathematical situations and in solving problems.

Finally, it is important that all involved with middle school mathematics instruction see that the content of Integrated Mathematics I and II is important both to life outside the classroom and to the continued success of students in their study of mathematics. The number sense, algebra, proportionality, and measurement concepts and skills in Integrated Mathematics I and II, combined with the understandings of data analysis and probability contained in these courses, are key to continued success in the study of mathematics and related disciplines and their application in real-world settings.

Integrated Mathematics I

STANDARDS

1. Nonnegative Rational Numbers and Concepts of Integers [Number and Operations][1]
2. Ratios and Rates [Number and Operations]
3. Two-Dimensional Geometry and Measurement [Geometry and Measurement]
4. Univariate Data Analysis [Data Analysis]
5. Experimental and Theoretical Probability [Probability]
6. Linear Patterns and Relationships [Algebra]

Course Goal:

Students compute fluently with nonnegative rational numbers and have facility with their application in mathematical and real-world settings. Students reason with rates and ratios to solve problems Students investigate, represent, and solve problems involving patterns, geometry, measurement, data, and probability.

Integrated Mathematics I corresponds closely to grade 6 in a traditional grade-level program. This course, together with Integrated Mathematics II, is designed to prepare students to take an integrated course containing topics from algebra and geometry as grade 8 students. The course focus is on completing the majority of work associated with nonnegative decimals and fractions. Integrated Mathematics I provides a solid conceptual base for the development of operations with integers and the consideration of the real numbers in Integrated Mathematics II. The geometry focus in the course is on the concepts of parallelism and perpendicularity and their use in describing and sketching familiar two-dimensional figures. Students investigate the effects of rigid transformations on figures in the coordinate plane, and they explore the symmetries of planar figures. Students' evolving knowledge of the integers allows them to represent and describe the locations of points in all four quadrants of the plane. Integrated Mathematics I students measure angles and find the perimeter and area of rectangles and triangles, and they convert measures within both the customary system and the metric system, but not between the systems.

Integrated Mathematics I students work on developing the ability to formulate a simple question about one small population, or about a comparison between two small populations, that can be answered by collecting and analyzing univariate data. They design straightforward data

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[1] The category mentioned in the brackets refers to corresponding content in the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (2000).

investigations, collect and analyze data, and use their analyses to answer the question they have formulated. Students develop abilities to describe the distribution of data values in such data sets (extremes, mean, median, mode, range, outliers) as well as to identify whether data are categorical or numerical. Students display collected data in graphs or tables, and they interpret the results of their investigation in the context of the formulated question.

Integrated Mathematics I students' experience with probability focuses on simple experiments viewed from experimental and theoretical standpoints. They develop a concept of probability as a ratio describing successes or failures to total trials and having a value between 0 and 1, inclusive.

In their study of content from algebra, these students examine and graph patterns showing linear relations, develop expressions with whole-number coefficients to represent linear patterns, and represent these patterns and relationships with graphs, tables, and manipulatives. Students learn how to evaluate linear expressions and solve one-step linear equations.

The following integrated mathematics standards and objectives do not represent any specific curricular design for an effective integrated mathematics instructional program. This integrated framework is provided to support curriculum and instructional designers as they connect and integrate this content into effective instruction.

Standard IMI.1: Nonnegative Rational Numbers and Concepts of Integers

Students develop number sense* related to nonnegative rational numbers, whole-number exponents, and integers. Students apply the concepts, properties, and operations associated with nonnegative rational numbers in solving multistep word problems. Students establish fluency in developing equivalent representations of and computing with these numbers as appropriate to given problem situations. Students apply models for the integers and additive inverses in developing an intuitive understanding of addition of integers.

Objective IMI.1.1:

Student develops number sense encompassing magnitude, comparison, order, and equivalent representations, which supports reasoning in operating with nonnegative rational numbers in fraction and decimal forms. Student applies these concepts, operations, and properties in solving problems involving relationships among whole numbers and other nonnegative rational numbers.

Performance Expectations:

- IMI.1.1.1 Represents, models, compares, and orders nonnegative rational numbers using graphical, pictorial, concrete, and numerical representations, including the use of equivalent fractions.
- IMI.1.1.2 Identifies and creates problem situations involving addition, subtraction, multiplication, and division of nonnegative rational numbers.
- IMI.1.1.3 Applies the properties associated with nonnegative rational numbers, including both their fraction and decimal representations, in solving problems.

Objective IMI.1.2:

Student becomes fluent in finding equivalent representations of, estimating, and computing with nonnegative rational numbers, in fraction and decimal forms, as appropriate to given problem situations.

Performance Expectations:

- IMI.1.2.1 Identifies and develops equivalent representations of nonnegative rational numbers, and translates fluently among these representations to fit a context or multistep problem situation, recognizing whether the numbers involved are reasonable.

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*Underlined words and phrases are defined in the Glossary.

IMI.1.2.2 Selects and applies an appropriate fraction or decimal representation for nonnegative rational numbers in a given context.

IMI.1.2.3 Selects an appropriate approach (estimate and check, mental math, paper-and-pencil, or technology) and applies it to solve a computational problem involving nonnegative rational numbers.

IMI.1.2.4 Computes fluently in situations involving addition, subtraction, multiplication, and division of nonnegative rational numbers in fraction and decimal forms.

IMI.1.2.5 Estimates results involving nonnegative rational numbers, and judges the reasonableness of both one's own and others' estimates.

Objective IMI.1.3:

Student develops a number sense related to natural-number exponents, including knowledge of powers, multiples, divisors, factors, primes, and composites. Student understands and applies the prime factorization of positive integers in solving problems.

Performance Expectations:

IMI.1.3.1 Reads, writes, and evaluates expressions involving natural-number powers of positive integers.

IMI.1.3.2 Makes observations, conjectures generalizations, and provides plausible explanations about numerical relationships, such as the divisibility rules for 2, 3, 5, 9, and 10.

IMI.1.3.3 Explains and applies the Fundamental Theorem of Arithmetic to represent numbers as products of prime factors.

IMI.1.3.4 Uses the prime factorization of two or three natural numbers to find their greatest common divisor/factor and least common multiple.

Objective IMI.1.4:

Student develops an understanding of the concepts of order, additive inverses, and addition of integers and solves simple addition problems involving integers.

Performance Expectations:

IMI.1.4.1 Describes the extension to the integers of concepts of order and position, and confirms the properties of addition of whole numbers (e.g., closure, associative, commutative, identity) for the addition of integers.

IMI.1.4.2 Recognizes and formulates problems involving the addition of integers, and solves them using number lines, patterns, and models.

IMI.1.4.3 Locates and plots points whose coordinates are both integers on the four-quadrant coordinate plane.

IMI.1.4.4 Recognizes and applies the absolute value of an integer as its distance from zero on a number line.

Standard IMI.2: Ratios and Rates

Students develop the ability to use ratios and rates to represent and reason about comparisons and to solve multistep problems, including those involving percent and those involving conversion of measures within the same measurement system.

Objective IMI.2.1:

Student identifies and represents ratios and rates as comparisons, and reasons to find equivalent ratios to solve problems involving proportional relationships.

Performance Expectations:

IMI.2.1.1 Identifies one or more ratios that represent a given comparison, and expresses the ratios using appropriate notation (i.e., $\frac{a}{b}$, a/b, a to b, $a : b$).

IMI.2.1.2 Describes the use of equivalent ratios in proportional relationships, and generates and applies appropriate equivalent ratios in problem situations involving scales and measurement conversion factors, percents, and probabilities.

Objective IMI.2.2:

Student develops computational fluency in working with ratios, percents, and proportional situations, and applies this fluency to estimate the solution to and solve a variety of real-world problems.

Performance Expectations:

IMI.2.2.1 Uses unit rates and equivalent ratios to represent and make quantitative comparisons in solving problems in real-life contexts (e.g., equivalent measures, discounts, interest, taxes, tips).

IMI.2.2.2 Represents and models ratios associated with whole-number percents that are less than or equal to 100%.

IMI.2.2.3 Mentally estimates and finds solutions to percent problems.

IMI.2.2.4 Translates fluently among the various representations for fractions, decimals, and percents in appropriate ways.

Standard IMI.3: Two-Dimensional Geometry and Measurement

Students develop and apply formulas for measurements in the plane, especially those involving the perimeter/circumference and area of triangles, quadrilaterals, circles, and composite figures made from these shapes. Students measure and classify angles and use this knowledge in representing and describing shapes and the results of rigid transformations of geometric figures on a coordinate plane.

Objective IMI.3.1:

Student distinguishes between length and area contexts, develops an understanding of formulas, and applies them to find the perimeter/circumference and area of triangles, quadrilaterals, circles, and composite figures made from these shapes.

Performance Expectations:

IMI.3.1.1 Distinguishes between appropriate units for linear and area measurement situations.

IMI.3.1.2 Describes the relationship between the circumference and diameter of a circle, $\pi = \frac{C}{d}$, and applies it to develop convincing arguments about the validity of formulas such as $C = 2\pi r$ and $A = \pi r^2$.

IMI.3.1.3 Finds the area of triangles, parallelograms, and trapezoids, by measuring or using measurement of sides and altitudes, by counting and rearranging portions of squares in a grid, and by developing and using formulas.

IMI.3.1.4 Uses given information to find the perimeter and area of a composite figure by dividing it into known shapes and using the measures of these shapes to determine the measures of the composite figure.

Objective IMI.3.2:

Student represents geometric figures from written or verbal descriptions, measurements, and properties using sketches, figures on grids, or models and makes conjectures concerning general properties and transformations of specified figures.

Performance Expectations:

IMI.3.2.1 Uses definitions and properties of two-dimensional figures to classify figures, and sketches figures having given properties.

IMI.3.2.2 Sketches two-dimensional figures with specified measures, both in rough sketch form and on graph paper.

IMI.3.2.3 Describes and applies the relationships of parallelism, perpendicularity, and symmetry in real-world settings involving two dimensions.

IMI.3.2.4 Identifies and graphs points in all four quadrants of the coordinate plane, and draws and labels the vertices of basic shapes on the plane.

IMI.3.2.5 Locates and gives the resulting transformed coordinates of a figure produced after a translation, a reflection about a vertical or horizontal line, or a rotation of a multiple of 90° about the origin of the coordinate plane.

IMI.3.2.6 Makes and supports conjectures about general properties of figures, such as the 180° angle sum for triangles, measures of the opposite angles in parallelograms, measures of vertical angles, and angle relationships in common shapes and figures.

Standard IMI.4: Univariate Data Analysis

Students formulate and answer questions about small populations by collecting and analyzing univariate data from the populations. Students draw conclusions from their analyses and communicate their results, and they identify flaws in faulty presentations of data found in the media.

Objective IMI.4.1:

Student formulates a question about one small population or about a comparison between two small populations that can be answered through data collection and analysis, designs related data investigations, and collects data.

Performance Expectations:

IMI.4.1.1 Formulates a simple question, and defines one or two small populations on which data can be collected to answer the question.

IMI.4.1.2 Identifies an attribute on which to collect data, decides how to measure the attribute to answer the question formulated, determines a data collection process, and collects the data.

IMI.4.1.3 Recognizes and describes the differences between numerical data and categorical data.

Objective IMI.4.2:

Student organizes and summarizes categorical and numerical data using summary statistics and a variety of graphical displays.

Performance Expectations:

IMI.4.2.1 Constructs appropriate graphical displays (bar graphs, line plots, stem-and-leaf plots, histograms), with and without technology, to describe the distribution of data values.

IMI.4.2.2 Describes the shape, center, and spread of the distribution of numerical data; constructs frequency distributions and determines the mode for categorical data.

IMI.4.2.3 Computes measures of center (mean, median) and spread (range) for a set of numerical data, with and without technology, interprets the meaning of these measures in context, and explains the influences of outliers on each measure.

Objective IMI.4.3:

Student interprets results and communicates conclusions regarding a formulated question using appropriate symbols, notation, and terminology. Student identifies flaws in faulty or misleading presentations of data found in the media.

Performance Expectations:

IMI.4.3.1 Interprets results and communicates conclusions in the context of the formulated question using appropriate symbols, notation, and terminology.

IMI.4.3.2 Identifies flaws in faulty or misleading presentations of data found in the media and their potential effects on conclusions drawn.

Standard IMI.5: Experimental and Theoretical Probability

Students estimate probabilities in experiments and, where possible, compare experimental and theoretical probabilities.

Objective IMI.5.1:

Student estimates probabilities in experiments with common objects, including comparing experimental and theoretical probabilities and examining experimental probabilities in the long run.

Performance Expectations:

IMI.5.1.1 Conducts experiments to estimate the likelihood of a simple event; compares the experimental probability with an easily identifiable theoretical probability, where the sample space is described by tables or lists of possible outcomes.

IMI.5.1.2 Describes why the probability of an event is a number between 0 and 1, inclusive.

IMI.5.1.3 Uses the probabilities of events to compare the likelihoods of events.

IMI.5.1.4 Uses experimental data with graphical or tabular displays to estimate the probability of an event for which the theoretical probability is unknown.

IMI.5.1.5 Recognizes that repetitions of an experiment may result in different outcomes, describes the variation in outcomes produced in an experiment, and recognizes that with the collection of more data the experimental probability of a particular outcome approaches the theoretical probability.

Standard IMI.6: Linear Patterns and Relationships

Students identify linear patterns for mathematical and real-world situations and represent linear expressions in words, tables, symbols, and graphs. Students write and solve one-step linear equations.

Objective IMI.6.1:

Student creates and evaluates simple linear expressions to represent linear patterns and develops graphs to represent these expressions.

Performance Expectations:

IMI.6.1.1 Represents linear patterns generated by mathematical and real-world situations with expressions, and evaluates these expressions for nonnegative rational numbers.

IMI.6.1.2 Generates and graphs a set of ordered pairs representing a given linear expression.

IMI.6.1.3 Writes rules in words and in symbols for situations modeled by the forms ax and $x \pm b$, where a and b are nonnegative rational numbers.

IMI.6.1.4 Extends linear patterns drawn from mathematical and real-world settings, sequences, tables, and graphs with verbal or symbolic rules of the form $ax \pm b$, where a and b are nonnegative rational numbers.

Objective IMI.6.2:

Student uses linear patterns to create simple linear equations, and student solves these equations.

Performance Expectations:

IMI.6.2.1 Uses linear patterns drawn from real-world settings to create equations and solve problems.

IMI.6.2.2 Evaluates nonnegative rational numbers as possible solutions to linear equations numerically and graphically, with and without technology.

IMI.6.2.3 Writes and solves linear equations of the form $ax = b$ and $x \pm b = c$, where a, b, and c and the solution are nonnegative rational numbers.

Integrated Mathematics II

STANDARDS

1. Integers and Rational Numbers [Numbers and Operations]
2. Two- and Three-Dimensional Geometry [Geometry]
3. Similarity and Measurement [Geometry and Measurement]
4. Bivariate Data [Data Analysis]
5. Probabilities in One-Stage Experiments [Probability]
6. Linear Equations and Inequalities [Algebra]

Course Goal:

Students reason with proportional relationships to structure and solve problems in two- and three-dimensional geometry and to connect rate of change, slope, linear relationships, and similarity on the coordinate plane. Students compute fluently with integers and rational numbers and apply them in solving problems, including those related to one-stage probability experiments.

Students in Integrated Mathematics II continue the transition from arithmetic to algebra. Students investigate and connect topics in number, geometry, measurement, probability, and algebra through the lens of proportionality. The focus on number changes from numbers and operations to systems and structures. Students develop a grasp of the system of integers and their operations, as well as integer exponents and their use in scientific notation. A significant amount of time is spent considering the notions of rates and ratios, and structuring and solving proportion problems, especially those involving direct variation. These topics are mirrored in geometry in the development of similar figures and the Pythagorean theorem, and the examination of and reasoning about geometric figures to develop properties and measurement formulas. These investigations allow a discussion of irrational numbers, such as π and $\sqrt{2}$, and the system of real numbers. Measurement work involves the extension of the concepts of perimeter and area in the plane to the study of surface area and volume and the related formulas for three-dimensional figures. Significant attention is given to the development of students' spatial visualization skills.

The data analysis processes in Integrated Mathematics II focus on situations where two pieces of data are collected from each member of a small population. Students develop their abilities to display and summarize bivariate data and to interpret and communicate their results. Students investigate simple one-stage experiments empirically and, where possible, compare theoretical and experimental probabilities.

Students learn to write algebraic expressions describing growth to deal with patterns that are linear (arithmetic) and to investigate the nature of growth in exponential (geometric) patterns. Specific emphasis is placed on developing students' abilities to represent linear relationships in a wide variety of formats, as well as interpreting the meaning of rate of change in different contexts. Within algebra, as well as across the other areas of Integrated Mathematics II, students have many opportunities to interpret contextualized situations using a number of different representations. These opportunities are studied explicitly, as well as being included within problem-solving situations. The capability to translate among representations encountered in pattern or data situations is central to the development of students' abilities to solve problems and to communicate results.

The following integrated mathematics standards and objectives do not represent any specific curricular design for an effective integrated mathematics instructional program. This integrated framework is provided to support curriculum and instructional designers as they connect and integrate this content into effective instruction.

Standard IMII.1: Integers and Rational Numbers

Students extend their number and operation sense to the entire set of integers and to the negative rational numbers, their varied representations, their operations, and their properties. Students develop fluency in computing with integers and rational numbers written in fraction and decimal forms. Students reason about and solve mathematical and real-world problems involving ratios, rates*, percents, and proportional relationships. Students develop an intuitive understanding of the real numbers as the set of all possible decimals, including irrationals such as π and $\sqrt{2}$.

Objective IMII.1.1:

Student models operations, computes fluently, and solves problems with integers.

Performance Expectations:

IMII.1.1.1 Models the addition, subtraction, multiplication, and division of integers, describes the relationships among these operations, and applies the order of operations.

IMII.1.1.2 Computes fluently with integers, including identifying roots of perfect squares and perfect cubes.

IMII.1.1.3 Represents and solves mathematical and real-world problems involving integers.

IMII.1.1.4 Estimates and judges the reasonableness of results involving integer operations.

Objective IMII.1.2:

Student computes fluently with rational numbers written in fraction and decimal forms, and student solves problems involving rational numbers.

Performance Expectations:

IMII.1.2.1 Describes and applies the relationships characterized by $a - b = a + (-b)$ and $a \div b = a \cdot \frac{1}{b}$.

IMII.1.2.2 Computes fluently with all rational numbers in both fraction and decimal form.

IMII.1.2.3 Represents and solves problems involving rational numbers, and judges the reasonableness of the solutions.

*Underlined words and phrases are defined in the Glossary.

Objective IMII.1.3:

Student describes the real numbers as the set of all decimal numbers and uses scientific notation, estimation, and properties of operations to represent and solve problems involving real numbers.

Performance Expectations:

IMII.1.3.1 Describes the real numbers as the set of all possible decimal numbers, and recognizes that representations of π, $\sqrt{2}$, and other irrational numbers are nonterminating, nonrepeating decimals.

IMII.1.3.2 Estimates results for mathematical and real-world problems involving operations and approximations with rational numbers, decimals, and percents, and judges the reasonableness of such computations and approximations.

IMII.1.3.3 Applies properties of real-number operations (associative, commutative, identity, inverse, distributive, closure, properties of equality and inequality) to solve a problem situation involving real numbers.

IMII.1.3.4 Reads, writes, and orders numbers represented in scientific notation using positive- and negative-integer exponents for powers of 10, and interprets applications of scientific notation in varied contexts including technological output formats.

Objective IMII.1.4:

Student reasons with ratios, rates, percents, and proportional relationships to solve problems and interpret results.

Performance Expectations:

IMII.1.4.1 Reasons about, structures, and solves problems involving rates, ratios, proportions, or percents, including percents less than 1 and greater than 100.

IMII.1.4.2 Interprets the meanings of rates of change associated with increases and decreases within mathematical and real-world contexts involving rates, ratios, measurement conversions within systems, scale drawings, and percents, and solves related problems.

IMII.1.4.3 Interprets and solves conversion problems involving scale drawings and constants of proportionality, and calculates derived measures such as density, foreign monetary equivalences, and those arising in other contexts.

Standard IMII.2: Two- and Three-Dimensional Geometry

Students formulate general statements defining and relating two- and three-dimensional figures by relevant characteristics and properties. Students recognize and apply properties of angles and figures, as well as the relationship between rigid transformations of figures and congruence of figures.

Objective IMII.2.1:

Student formulates general statements relating two- and three-dimensional figures using their relevant characteristics and geometric properties.

Performance Expectations:

IMII.2.1.1 Formulates general statements describing properties of circles, polygons, prisms, pyramids, cones, spheres, and right-circular cylinders.

IMII.2.1.2 Relates and applies planar nets in analyzing and representing three-dimensional figures in terms of related two-dimensional figures.

IMII.2.1.3 Represents three-dimensional figures by sketches and by using isometric dot paper.

Objective IMII.2.2:

Student identifies, justifies, and applies angle relationships in describing geometric figures and relationships.

Performance Expectations:

IMII.2.2.1 Reasons using models and drawings to construct and support convincing arguments about angle relationships in figures.

IMII.2.2.2 Identifies, states, and applies the basic properties associated with complementary angles and angles formed by transversals intersecting pairs of parallel lines.

IMII.2.2.3 Identifies, states, and applies the angle-sum properties for triangles and other polygons.

Objective IMII.2.3:

Student relates and applies knowledge of rigid transformations.

Performance Expectations:

IMII.2.3.1 Describes the effects of rigid transformations (translations, reflections about a vertical or horizontal line, rotations about the origin, and simple compositions of these transformations) on figures in the coordinate plane.

IMII.2.3.2 Uses rigid transformations to identify the corresponding parts of congruent figures.

Standard IMII.3: Similarity and Measurement

Students develop and apply similarity relationships, the Pythagorean theorem, and other indirect measurement methods to solve problems. Students apply the concepts of surface area and volume to measure three-dimensional figures.

Objective IMII.3.1:

Student identifies, describes, and applies similarity relationships to find measures of corresponding parts in similar figures and applies scales to measurements in drawings and maps.

Performance Expectations:

IMII.3.1.1 Defines and identifies similarity of two-dimensional figures, including the linking of corresponding parts, the similarity ratio, and the measures of corresponding parts.

IMII.3.1.2 Determines a proportional relationship among measures of corresponding sides of similar figures.

IMII.3.1.3 Solves indirect measurement problems and scaling problems involving mathematical and real-world contexts using similar figures, with or without grids.

IMII.3.1.4 Interprets and solves situations using scales, including those based on number lines, drawings, models, maps, and graphs.

Objective IMII.3.2:

Student develops and applies the Pythagorean theorem to solve measurement problems.

Performance Expectations:

IMII.3.2.1 Develops the Pythagorean theorem by investigating right triangles, their measures, and related areas.

IMII.3.2.2 Applies the Pythagorean theorem to solve measurement problems.

Objective IMII.3.3:

Student applies the concepts of surface area and volume to measure three-dimensional figures.

Performance Expectations:

IMII.3.3.1 Investigates and describes the relationships between the measurements of three-dimensional figures and the measures of related two-dimensional figures (e.g., $A = l \cdot w$, $V = (l \cdot w) \cdot h$).

IMII.3.3.2 Investigates, conjectures, and applies formulas for determining the surface areas and volumes of rectangular solids, other prisms, and cylinders.

IMII.3.3.3 Formulates and applies general statements relating scale changes in the linear dimensions of a figure to the resulting changes in perimeter, area, circumference, surface area, and volume of the resulting figure.

Standard IMII.4: Bivariate Data

Students formulate and answer questions about a small population by collecting, organizing, and analyzing bivariate data from the population. Students summarize and represent bivariate data. Students communicate the results of data analyses and identify flaws in faulty or misleading presentations of bivariate data found in the media.

Objective IMII.4.1:

Student formulates questions about a small population that can be answered through collection and analysis of bivariate data, designs related data investigations, and collects data.

Performance Expectations:

IMII.4.1.1 Formulates a simple question involving two attributes, and defines a small population on which data can be collected to answer the question.

IMII.4.1.2 Identifies two attributes on which to collect data, decides how to measure the two attributes to answer the question formulated, determines a data collection process, and collects data.

Objective IMII.4.2:

Student organizes and summarizes bivariate data, examining data on the two attributes separately and together, and classifies each attribute as a categorical variable or numerical variable. Student uses summary statistics and a variety of graphical displays to represent the data.

Performance Expectations:

IMII.4.2.1 Classifies each attribute as corresponding to a categorical or numerical variable. Describes the distribution for each attribute separately using appropriate graphs, including stem-and-leaf plots, histograms, and box plots, and summary statistics, including interquartile range.

IMII.4.2.2 Identifies, describes, and constructs appropriate displays for bivariate data: two-way tables for two categorical variables; parallel box plots or back-to-back stem-and-leaf plots for one numerical and one categorical variable; and scatterplots, with an appropriately sketched trend line, for two numerical variables.

IMII.4.2.3 Explains the usefulness of different displays for bivariate data. Describes the relationship between the two variables and the effects of outliers on the observed relationship.

Objective IMII.4.3:

Student interprets results and communicates conclusions from a bivariate data analysis to answer the formulated question using appropriate symbols, notation, and terminology. Student identifies flaws in faulty or misleading presentations of bivariate data found in the media.

Performance Expectations:

IMII.4.3.1 Interprets results and communicates conclusions from a bivariate data analysis in the context of the formulated question using appropriate symbols, notation, and terminology.

IMII.4.3.2 Identifies flaws in faulty or misleading presentations of bivariate data found in the media and their potential effects on conclusions drawn.

Standard IMII.5: Probabilities in One-Stage Experiments

Students determine the sample space for one-stage experiments and determine the experimental and, where possible, theoretical probabilities for events defined on the sample space. Students describe probabilities of events governed by the addition rule for probabilities.

Objective IMII.5.1:

Student determines the sample space for one-stage experiments and determines, where possible, the theoretical probabilities for events defined on the sample space. Student describes and applies the addition rule for probabilities.

Performance Expectations:

IMII.5.1.1 Determines the sample space for a given one-stage experiment, and uses lists, tables, and tree diagrams to represent all possible outcomes.

IMII.5.1.2 Identifies events for a given sample space, represents relationships among events using Venn diagrams, and determines the probabilities of events and their complements.

IMII.5.1.3 Describes and applies the addition rule for probabilities for events that are mutually exclusive and for events that are not.

Standard IMII.6: Linear Equations and Inequalities

Students develop and apply the connections between rate of change and linear relationships. They create and solve linear equations using tabular and graphical displays, verbal representations of problems, and symbolic manipulation. Students interpret inequalities involving one variable.

Objective IMII.6.1:

Student interprets rate of change in real-world and in mathematical settings and recognizes the constant rate of change associated with linear relationships.

Performance Expectations:

IMII.6.1.1 Demonstrates that the rate of change in linear settings is constant, and graphically describes the proportional relationship embedded in this rate of change and represented in the "steepness" or inclination of the related line.

IMII.6.1.2 Interprets, describes, and uses rates of change to model mathematical and real-world situations, noting the different patterns resulting from linear (arithmetic) and exponential (geometric) patterns.

IMII.6.1.3 Connects and translates among equivalent representations of linear relationships, including coordinate graphs and their related tables of values, equations, and verbal rules, to solve problems involving linear patterns.

Objective IMII.6.2:

Student creates one- and two-step linear equations and solves such equations using tables, coordinate graphs, and symbolic manipulation.

Performance Expectations:

IMII.6.2.1 Represents linear mathematical and real-world situations using a linear equation of the form $ax + b = c$, where a, b, and c are rational numbers expressed as fractions, decimals, or integers.

IMII.6.2.2 Solves linear equations with rational-number coefficients using mental, graphical, and symbolic methods, with and without technology.

IMII.6.2.3 Connects the graphical, tabular, and symbolic representations of the unique solution of a given linear equation.

Objective IMII.6.3:

Student represents and interprets inequalities in one variable geometrically and symbolically.

Performance Expectations:

IMII.6.3.1 Represents solutions to inequalities such as $x > a$ and $a \leq x \leq b$ on a number line.

IMII.6.3.2 Writes an inequality to represent an interval or ray, with or without endpoints, shown on the number line.

Integrated Mathematics III Through Integrated Mathematics VI

The most important thing that students learn in Integrated Mathematics III through VI is the existence of rich connections among number systems, algebra, geometry and measurement, probability, and statistics as they are used to model the world and solve problems. Students learn to view mathematics as a way of understanding the world about them and how mathematics allows one to predict and even control outcomes in a variety of applications. The content of these courses provides a foundation for the further study of mathematics and related subjects, for career paths that make significant use of mathematics, and for solving problems and making decisions throughout students' adult lives.

These courses also complete the transition of students' study of mathematics from the more intuitive number- and shape-based experiences of the middle grades to a study centered on functions, translations, structures, and statistical models. In these courses, students encounter functions and learn to compare and contrast the properties of families of linear, quadratic, general polynomial, exponential, logarithmic, and trigonometric functions. They apply these functions to study and model phenomena as well as to predict future occurrences. Students' sense of justification and reasoning grows as they come to see the roles that axiomatic structures play in placing mathematical thought on a firm foundation. They learn new modes of reasoning and ways to form conjectures and test their validity. They see the fundamental roles played by approaches based in synthetic geometry* and in coordinate geometry to studying congruence, similarity, and rigid motions in the plane.

Students mature in their understanding of probability through the study of counting and ways of determining or estimating probabilities of events via theoretical or empirical methods. Their knowledge of statistics grows through their experiences with surveys, experiments, and observational studies. Along the way, they develop a solid understanding of the role that randomization and design play in the collection and interpretation of data.

As students continue to grow in their understanding of mathematics and statistics, they should also mature in their flexible use of technology tools—graphing calculators, spreadsheets, computer algebra systems (CAS), dynamic geometry software, and statistical software.

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*Underlined words and phrases are defined in the Glossary.

Integrated Mathematics III

STANDARDS

1. Patterns of Change and Algebraic Representations [Algebra]
2. Variables, Expressions, Equations, and Functions in Linear Settings [Algebra]
3. Exponential and Quadratic Equations and Functions [Algebra]
4. Patterns of Shape, Geometric Reasoning, and Geometric Relationships [Geometry and Measurement]
5. Surveys and Random Sampling [Data Analysis]

Course Goal:

Students reason within situations involving change and represent change in the form of expressions, equations, inequalities, and functions to model and solve problems in linear, exponential, and quadratic settings. Students examine the validity of geometric reasoning in forming and evaluating arguments about two- and three-dimensional relationships within the structure of an axiomatic system and in investigating situations represented by edge-vertex graphs. Students reason statistically to formulate and answer questions using surveys.

Integrated Mathematics III is designed to provide students with a comprehensive view of linear relationships from both an equation- and a function-based standpoint. This study involves students in using mathematics within a variety of settings and constantly transforming problem representations from geometric-spatial settings to algebraic-numerical representations. Students' understanding of algebra grows from numerical settings to focus on the use of variables to describe, investigate, represent, and solve problems involving linear relationships. Students extend this knowledge to develop strategies for conceptualizing, representing, and solving problems involving exponential and quadratic relationships. Throughout this study, the focus is examining the nature and representation of change as it appears in these settings. In this way, Integrated Mathematics III provides a bridge from the study of patterns and relations in previous grades to the study of linear, exponential, and quadratic functions. In Integrated Mathematics III, functions are represented by tables, graphs, symbolic and verbal expressions, and formulas. These functions are applied in analyzing data from other disciplines and in extending students' knowledge of representing and measuring geometric objects in their environment. Learning experiences support the development of the ability to distinguish between linear and nonlinear phenomena found in the study of biology, chemistry, business, and other disciplines.

Symbolically, students recognize that there are many different—but equivalent—representations of an expression, function, or equation and that these representations differ in their efficiency in interpreting or solving a problem depending on the context. Students become fluent in translating among different representations of linear situations, as well as between different forms of the same type of representation.

Their introduction to and exploration of exponential and quadratic expressions allows students to better understand constant rate of change as a defining feature of linearity and to develop foundations for understanding and interpreting the types of change found in exponential and linear settings. Integrated Mathematics III students also explore and develop efficient ways of solving and estimating solutions to exponential and quadratic equations, including completing the square to solve quadratic equations. However, the formal development of the quadratic formula is left to Integrated Mathematics V, when its use is linked to both the complex numbers and more-general features of parabolic functions.

Students should have a wide range of experiences interpreting geometric objects and their representations, as well as activities in which they have to develop or explain their own representations for geometric entities. These experiences provide students with a language for describing the world about them in terms of geometric objects and relationships. The study of geometric concepts and relationships in Integrated Mathematics III focuses on extending students' previous experiences with forming conjectures and on developing convincing arguments to support them. While deductive methods are used in number and algebra work as well, the focus on reasoning, justification, and proof is often more visible in geometry. Students develop short arguments supporting statements about relationships of lines and angle measures, as well as test the validity of proposed arguments. Students begin to develop an understanding of the nature of a mathematical system and the role that proof plays in justifying conjectures.

Data analysis focuses on the collection and analysis of survey data. Students learn to draw a random sample from a population, and to distinguish between random and nonrandom methods for drawing samples and between sampling error and measurement error. Students learn to identify sources of bias in sampling and apply their knowledge of sampling to make judgments about the quality of inferences that can be drawn from different types of samples. The planning and conducting of data collection, and representation and interpretation of data all involve other aspects of students' work with number, geometry and measurement, and algebra—thus integrating the major concepts and skills studied within the year's work.

The following integrated mathematics standards and objectives do not represent any specific curricular design for an effective integrated mathematics instructional program. This integrated framework is provided to support curriculum and instructional designers as they connect and integrate this content into effective instruction.

Standard IMIII.1: Patterns of Change and Algebraic Representations

Students use constant rate of change to identify situations that can be represented by linear* functions, model these situations, interpret the models, and solve mathematical and real-world problems. Students use rate of change and graphical representations to contrast linear relationships with nonlinear relationships.

Objective IMIII.1.1:

Student identifies functions based on their graphical behavior and rates of change, and student describes functions using appropriate notation and terminology.

Performance Expectations:

IMIII.1.1.1	Determines whether a relationship is a function by using a graph or a verbal description of the relationship.
IMIII.1.1.2	Determines whether a relationship is linear or nonlinear based on whether it has a constant rate of change, its verbal description, its table of values, its graphical representation, or its symbolic form.
IMIII.1.1.3	Describes characteristics of piecewise-linear functions, including absolute value, and situations in which they arise.
IMIII.1.1.4	Applies the terminology and symbols associated with expressions, functions, and linear equations, including function notation, inputs, outputs, domain, range, slope, intercepts, and independent and dependent variables.

Objective IMIII.1.2:

Student uses linear functions to interpret, model, and solve situations having a constant rate of change.

Performance Expectations:

IMIII.1.2.1	Generalizes linear patterns or arithmetic sequences using verbal rules and symbolic expressions such as kx and $ax + b$ in representing proportional or more-general linear relationships, respectively.
IMIII.1.2.2	Analyzes a mathematical or real-world situation; determines whether the situation can be described by a linear model, and if so, determines the constant rate of change and develops and interprets a linear function to model that situation.

*Underlined words and phrases are defined in the Glossary.

Standard IMIII.2: Variables, Expressions, Equations, and Functions in Linear Settings

Students distinguish among different uses of variables and find equivalent expressions and equations. They construct, represent, solve, and interpret solutions of linear equations and linear inequalities for mathematical and real-world contexts.

Objective IMIII.2.1:

Student represents linear patterns using expressions, equations, functions, and inequalities and interprets the meanings of these representations, recognizing which are equivalent and which are not.

Performance Expectations:

IMIII.2.1.1 Represents linear patterns using tables, graphs, sequences, verbal rules, symbolic expressions, equations, and functions of the form $f(x) = ax + b$.

IMIII.2.1.2 Describes the meaning of symbolic expressions of the form $ax + b$ in words, and interprets the changes resulting from different values of the parameters a and b.

IMIII.2.1.3 Develops equivalent algebraic expressions, equations, and inequalities using the properties of equality and inequality, as well as the commutative, associative, inverse, identity, and distributive properties.

IMIII.2.1.4 Identifies and translates among equivalent representations of linear expressions, equations, inequalities, and systems of equations, using verbal, tabular, graphical, and symbolic representations.

IMIII.2.1.5 Writes, interprets, and translates among equivalent forms of linear equations and functions, including slope-intercept, point-slope, intercept, and general forms, recognizing that equivalent forms for a linear relationship reveal more or less information about a given situation.

Objective IMIII.2.2:

Student distinguishes among the different uses of variables, parameters, constants, and equations.

Performance Expectations:

IMIII.2.2.1 Describes and distinguishes among the different uses of variables: as symbols for varying quantities (such as $5x$); as symbols for fixed and possibly unknown values in equations (such as $3x + 2 = 5$); as symbols for all numbers in properties (such as $x + x = 2x$); as symbols in formulas (such as $A = l \cdot w$); and as symbols for parameters (such as the m for slope in $y = mx + b$).

IMIII.2.2.2 Identifies the constant and variable terms in linear expressions, equations, and inequalities and in systems of equations and inequalities.

IMIII.2.2.3 Identifies and distinguishes among parameters and the independent and dependent variables in a linear relationship (e.g., in $y = mx + b$, x and y are the independent and dependent variables, respectively, and m and b are the parameters).

IMIII.2.2.4 Describes and distinguishes among the types of equations that can be constructed by equating linear expressions, including identities (e.g., $x + x = 2x$); equations for which there is no solution (e.g., $x + 1 = x + 2$); formulas (e.g., $C = \pi d$); equations where the solution is unique (e.g., $2x + 3 = 5$); and equations relating two variables (e.g., $y = 3x + 7$).

Objective IMIII.2.3:

Student constructs, solves, and interprets solutions of linear equations and linear inequalities representing mathematical and real-world contexts.

Performance Expectations:

IMIII.2.3.1 Constructs a linear equation or linear inequality to model a real-world situation, using a variety of methods and representations.

IMIII.2.3.2 Analyzes and explains the reasoning used to solve linear equations and linear inequalities.

IMIII.2.3.3 Solves a linear equation or inequality using symbolic methods, graphs, tables, and technology.

Standard IMIII.3: Exponential and Quadratic Equations and Functions

Students identify and classify exponential and quadratic relationships. Students represent simple mathematical and real-world phenomena using exponential and quadratic functions and solve equations involving these functions with a variety of techniques.

Objective IMIII.3.1:

Student identifies certain nonlinear relationships and classifies them as exponential relationships or quadratic relationships, including relationships of the form $y = \frac{k}{x}$, based on rates of change in tables, symbolic forms, or graphical representations. Student recognizes that multiplying linear factors produces nonlinear relationships.

Performance Expectations:

IMIII.3.1.1 Identifies nonlinear (exponential, quadratic, and equations of the form $y = \frac{k}{x}$) relationships in graphical or tabular displays through an examination of successive differences, ratios, symbolic forms, or graphical properties.

IMIII.3.1.2 Identifies terms in a geometric (exponential) sequence using verbal rules or symbolic expressions.

IMIII.3.1.3 Multiplies a pair of linear expressions, and interprets the result of the operation numerically by evaluation, through a table of values, and graphically.

Objective IMIII.3.2:

Student represents and interprets simple exponential and quadratic functions based on mathematical and real-world phenomena using tables, symbolic forms, or graphical representations and solves equations related to these functions.

Performance Expectations:

IMIII.3.2.1 Finds integer powers of rational numbers; evaluates the meaning of integer powers of variables in expressions, and applies the basic laws of exponents ($a^m \cdot a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, and $(ab)^n = a^n b^n$ and for all $a \neq 0$, $a^0 = 1$ and $\frac{a^m}{a^n} = a^{m-n}$).

IMIII.3.2.2 Recognizes exponential functions from their verbal description and tabular, graphical, or symbolic representations, and translates among these representations.

IMIII.3.2.3 Describes the effects of changes in the coefficient, base, and exponent on the growth described by an exponential function.

IMIII.3.2.4 Distinguishes among general representations for exponential equations ($y = b^x$, $y = a(b^x)$) and quadratic equations ($y = x^2$, $y = -x^2$, $y = ax^2$, $y = x^2 + c$, $y = ax^2 + c$), and describes how the values of a, b, and c affect their graphical and tabular representations.

IMIII.3.2.5 Provides and describes multiple representations of solutions to simple exponential and quadratic equations using manipulative models, tables, graphs, and symbolic expressions, and using technology.

IMIII.3.2.6 Factors simple quadratic expressions (limited to the removal of monomial terms, perfect-square trinomials, difference of squares, and quadratics of the form $x^2 + bx + c$ that factor over the integers), and applies the zero-product property to determine the solutions of the related equation.

IMIII.3.2.7 Solves quadratic equations using completing the square and technology, and interprets such solutions in terms of the original problem context.

Standard IMIII.4: Patterns of Shape, Geometric Reasoning, and Geometric Relationships

Students represent geometric objects and investigate a variety of relationships among them, form conjectures, and attempt to verify or reject the conjectures. Students develop and apply various methods of proving statements or disproving conjectures within the axiomatic structure of Euclidean geometry.

Objective IMIII.4.1:

Student uses a variety of representations to describe geometric objects and to analyze relationships among them. Student examines elementary models of non-Euclidean geometries and finite geometries to understand the nature of axiomatic systems and the role the parallel postulate plays in shaping Euclidean geometry.

Performance Expectations:

IMIII.4.1.1 Uses coordinates and algebraic representations (e.g., distances, points that divide segments in specified ratios, slope) to describe and define figures.

IMIII.4.1.2 Uses nets, drawings, models, and technologically created images to represent geometric objects and analyze relationships among them.

IMIII.4.1.3 Investigates geometric representations and properties not found in Euclidean geometry, for example, relationships from geometry on a sphere or applications of planar networks.

IMIII.4.1.4 Interprets the role of the parallel postulate as the key postulate in the formulation of Euclidean geometry, and illustrates its counterparts in other geometries (e.g., geometry on a sphere, in a finite geometry).

Objective IMIII.4.2:

Student develops, tests, and provides justifications, based on inductive and deductive methods, for conjectures involving relations of lines, angles, and figures.

Performance Expectations:

IMIII.4.2.1 Describes the structure of and relationships within an axiomatic system (undefined terms, defined terms, axioms/postulates, methods of reasoning, and theorems).

IMIII.4.2.2 Recognizes flaws or gaps in the reasoning supporting an argument.

IMIII.4.2.3 Develops and tests conjectures about angles, lines, bisectors, polygons (especially triangles and quadrilaterals), circles, and three-dimensional figures.

IMIII.4.2.4 Justifies statements about angles formed by perpendicular lines and transversals of parallel lines.

Standard IMIII.5: Surveys and Random Sampling

Students design and conduct a survey based on an appropriate sample, and they interpret and communicate the results. Students evaluate survey results reported in the media. Students discern the differences between random and nonrandom sampling methods and can identify sources of bias in sampling. Students distinguish between sampling error and measurement error, and they understand that results may vary from sample to population and from sample to sample.

Objective IMIII.5.1:

Student formulates questions that can be addressed through collection and analysis of survey data. Student explains the importance of random selection of members from the population, and designs and executes surveys. Student uses the results of a survey to communicate an answer appropriate to the question of interest. Student distinguishes between sampling error and measurement error. Student evaluates survey results reported in the media.

Performance Expectations:

IMIII.5.1.1 Formulates a question of interest and defines key components that can be addressed through a survey. Defines the population, the variables to measure, and how to measure the variables; identifies factors that may influence survey outcomes; designs questionnaires.

IMIII.5.1.2 Describes techniques for drawing simple random samples of members from a population. Identifies situations in which a stratified random sample from a population would be preferred over a simple random sample.

IMIII.5.1.3 Identifies and describes the differences between a sample and a census, explaining the advantages and disadvantages of each.

IMIII.5.1.4 Designs and implements the selection of a simple random sample from a population; collects and organizes survey data; displays the data in appropriate tables or graphs; and summarizes the data using measures of center and spread, including the mean absolute deviation.

IMIII.5.1.5 Explains the question of interest, the sampling methods used to answer the question, and the results obtained in the context of the question.

IMIII.5.1.6 Describes how the method of selecting subjects for a sample and the methods of measurement of outcomes can affect survey results. Explains how biases may arise from both sampling errors and measurement errors.

IMIII.5.1.7 Examines survey results reported in the media, discussing and evaluating how the sample was drawn from the population and the methods used to measure, collect, and represent the data collected. Identifies possible sources of bias that may affect survey results.

Objective IMIII.5.2:

Student understands that results may vary from sample to population and from sample to sample. Student analyzes, summarizes, and compares results from random and nonrandom samples and from a census, using summary numbers and a variety of graphical displays to communicate findings.

Performance Expectations:

IMIII.5.2.1 Compares measures of center and spread computed using sample data drawn from a population (sample statistics) with the same measures of center and spread computed using data from a census of the population (population parameters). Observes that sample means tend to approach the population mean as sample size increases.

IMIII.5.2.2 Recognizes that measures of center and spread computed using data from a random sample are likely to differ from sample to sample even when the samples are drawn from the same population and have the same number of observations.

IMIII.5.2.3 Distinguishes between random and nonrandom sampling methods. Compares results from simple random and nonrandom samples drawn from the same population; discusses how and why the results might differ because of potential sources of bias in the various samples.

Integrated Mathematics IV

STANDARDS

1. Systems of Linear Equations and Matrices [Algebra]
2. Geometric Proof, Similarity, and Transformations [Geometry]
3. Direct and Indirect Measurements [Geometry and Measurement]
4. Two-Stage Experiments, Conditional Probability, and Independence [Probability]
5. Bivariate Data and Trend Line Models [Data Analysis]

Course Goal:

Students structure and solve systems of linear equations, and they learn to use matrix methods. They formalize their understanding of congruence and similarity and prove theorems concerning these relationships for planar figures. Students reason to solve problems in two- and three-dimensional geometry and measurement. Students reason through two-stage probability models to represent, interpret, and discuss situations involving conditional probability. They represent bivariate data, model the data with trend lines, and measure the linear association between two variables with the correlation coefficient.

Students reason to structure and solve problems that can be modeled by systems of linear equations using a variety of approaches, including matrix methods. As students extend their study of geometric proof, they encounter formal definitions of previously developed concepts. They work with postulates and these definitions to construct a more formal framework that supports the forming and testing of conjectures. Students focus on the use of geometric properties and relationships in understanding spatial settings and solving problems, defining and using algebraic relationships to model transformations, solving geometric problems in coordinate systems, and applying measurement properties to find distances and make indirect measurements in two and three dimensions.

Students' knowledge of probability is extended to the understanding of chance in two-stage experiments, when the stages are independent and when they are not. Simulation is used as a tool for estimating probabilities for events where theoretical values are difficult or impossible to compute. Students develop and extend their knowledge of bivariate data using scatterplots and median-fit lines to model trends observed in data. This work makes use of a significant amount of students' knowledge and skills from algebra and the study of functions.

The content areas of number and operations, data analysis and probability, and algebra are constantly linked to the study of geometry and measurement through the viewing of problems from a geometric-spatial viewpoint and the use of coordinates to measure and describe geometric situations.

The following integrated mathematics standards and objectives do not represent any specific curricular design for an effective integrated mathematics instructional program. This integrated framework is provided to support curriculum and instructional designers as they connect and integrate this content into effective instruction

Standard IMIV.1: Systems of Linear Equations and Matrices

Students represent relationships that can be modeled by a system of linear* equations and solve the system using a variety of methods and representations. They represent and interpret data and systems of equations through matrix representations, using addition and multiplication of matrices as appropriate. Students use matrix equations and inverses, where they exist, to find solutions to systems of equations using technology.

Objective IMIV.1.1:

Student represents relationships that can be modeled by a system of linear equations and inequalities and solves the system using a variety of methods and representations.

Performance Expectations:

IMIV.1.1.1 Constructs a system of linear equations modeling a real-world situation, using a variety of methods and representations.

IMIV.1.1.2 Analyzes and explains the reasoning used to solve a system of linear equations.

IMIV.1.1.3 Solves a system consisting of two linear equations in two unknowns, using graphs, tables, symbolic methods, and technology, and describes the nature of the solutions (no solution, one solution, infinitely many solutions).

IMIV.1.1.4 Solves the equation $r = ax + b$ by using the fact that the value of x determined by this equation is the x-coordinate of the solution to the system of equations $\left\{\begin{matrix} y = ax + b \\ y = r \end{matrix}\right\}$. Relates this solution method to graphical and technology-aided methods of solving equations.

Objective IMIV.1.2:

Student represents and interprets cross-categorized data in matrices, develops properties of matrix addition, and uses matrix addition and its properties to solve problems.

Performance Expectations:

IMIV.1.2.1 Represents numerical or relational data categorized by two variables in a matrix and labels the rows and columns. Interprets the meaning of a particular entry in a matrix in terms of the labels of its row and column.

IMIV.1.2.2 Uses matrix row and column sums to analyze data.

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*Underlined words and phrases are defined in the Glossary.

IMIV.1.2.3 Develops the properties of matrix addition, and adds and subtracts matrices to solve problems.

Objective IMIV.1.3:

Student multiples matrices, verifies the properties of matrix multiplication, and uses the matrix form for a system of linear equations to structure and solve systems consisting of two or three linear equations in two or three unknowns, respectively, with technology.

Performance Expectations:

IMIV.1.3.1 Verifies the properties of matrix multiplication and multiplies matrices to solve problems.

IMIV.1.3.2 Constructs a system of linear equations modeling a real-world situation, and represents the system as a matrix equation ($A\boldsymbol{x} = \mathbf{b}$), that is,

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

IMIV.1.3.3 Solves a system consisting of two or three linear equations in two or three unknowns, respectively, by solving the related matrix equation $A\boldsymbol{x} = \mathbf{b}$, using technology to find $\boldsymbol{x} = A^{-1}\mathbf{b}$.

Standard IMIV.2: Geometric Proof, Similarity, and Transformations

Students develop general methods of proof and apply these methods to solve problems involving congruence, similarity, rigid transformations, and origin-centered dilations of figures in the plane.

Objective IMIV.2.1:

Student applies mathematical methods of proof to develop justifications for basic theorems of Euclidean geometry.

Performance Expectations:

IMIV.2.1.1 Forms conjectures based on exploring geometric situations with or without technology.

IMIV.2.1.2 Proves, directly or indirectly, that a valid mathematical statement is true. Develops a counterexample to refute an invalid statement.

IMIV.2.1.3 Formulates and investigates the validity of the converse of a conditional statement.

IMIV.2.1.4 Organizes and presents direct proofs and indirect proofs using two-column, paragraph, and flow-chart formats.

Objective IMIV.2.2:

Student identifies congruent figures and justifies these congruences by establishing sufficient conditions and by finding a congruence-preserving transformation between the figures. Student solves problems involving congruence in a variety of contexts.

Performance Expectations:

IMIV.2.2.1	Analyzes figures in terms of their symmetries using the concepts of reflection, rotation, and translation and combinations of these.
IMIV.2.2.2	Compares and contrasts equality, congruence, and similarity.
IMIV.2.2.3	Identifies and differentiates among sufficient conditions for congruence of triangles (SSS, SAS, ASA, AAS, and HL), and applies them.
IMIV.2.2.4	Uses coordinate geometry and rigid transformations (reflections, translations, and rotations) to establish congruence of figures.

Objective IMIV.2.3:

Student identifies and applies transformations of figures in the coordinate plane and discusses the results of these transformations.

Performance Expectations:

IMIV.2.3.1	Represents translations, line reflections, rotations, and origin-centered dilations of objects in the coordinate plane by using sketches, coordinates, and function notation, and explains the effects of these transformations.
IMIV.2.3.2	Recognizes and identifies corresponding parts of congruent and similar figures after transformation.

Objective IMIV.2.4:

Student identifies similar figures and justifies these similarities by establishing sufficient conditions and by finding a similarity-preserving rigid transformation or origin-centered dilation between the figures. Student solves problems involving similarity in a variety of contexts.

Performance Expectations:

IMIV.2.4.1	Identifies the similarity conditions SAS, SSS, and AA as sufficient conditions for establishing similarity of triangles, and applies them, noting that congruence is a special case of similarity.
IMIV.2.4.2	Uses similarity to calculate the measures of corresponding parts of similar figures, and applies similarity in a variety of problem-solving contexts within mathematics and other disciplines.

IMIV.2.4.3	Creates a representation of a figure similar to a specified figure given their similarity ratio.
IMIV.2.4.4	Uses similar triangles to demonstrate that the rate of change associated with any pair of points on a line is the same.
IMIV.2.4.5	Uses origin-centered dilations to describe and investigate similarities.

Standard IMIV.3: Direct and Indirect Measurements

Students justify and apply the measurement formulas associated with one-, two-, and three-dimensional geometric objects.

Objective IMIV.3.1:

Student justifies two- and three-dimensional measurement formulas for perimeter/circumference, area, and volume and applies these formulas and other geometric properties relating angle and arc measures to solving problems involving measures of simple and composite one-, two-, and three-dimensional geometric objects.

Performance Expectations:

IMIV.3.1.1	Justifies the area formulas for quadrilaterals and regular polygons.
IMIV.3.1.2	Applies the (volume) = (area of the base) × (height) principle in linking area and volume formulas for prisms and cylinders.
IMIV.3.1.3	Links the surface area of prisms and cylinders to the sum of the areas of their bases and lateral surfaces using planar nets to illustrate and sum the relevant measures.
IMIV.3.1.4	Identifies and finds the measures of angles formed by segments in three-dimensional figures, extending right-triangle and isosceles/equilateral-triangle relationships to study the planar faces of three-dimensional objects.
IMIV.3.1.5	Applies formulas and solves problems involving area, perimeter, volume, and surface area of pyramids, cones, spheres, and composite figures.
IMIV.3.1.6	Determines arc lengths of circles and areas of sectors of circles using proportions.
IMIV.3.1.7	Develops the triangle angle-sum and angle-measure theorems for polygons, and the triangle- and angle-inequality theorems.
IMIV.3.1.8	Justifies and applies statements about angles formed by chords, tangents, and secants in circles and the measures of their intercepted arcs.

Standard IMIV.4: Two-Stage Experiments, Conditional Probability, and Independence

Students determine the sample space for two-stage experiments. They distinguish between independent and dependent events and compute their probabilities. Students use simulation to solve real-world probability problems.

Objective IMIV.4.1:

Student determines the sample space for two-stage experiments, and employs the multiplication rule for counting (Fundamental Property of Counting). Student distinguishes between independent and dependent compound events and computes their probabilities using representations for such events and using the multiplication rule for probability.

Performance Expectations:

- IMIV.4.1.1 Uses lists, tables, and tree diagrams to represent all possible outcomes in the sample space for a given two-stage experiment.
- IMIV.4.1.2 Employs systematic counting approaches, such as the multiplication rule for counting (Fundamental Property of Counting), to determine the number of possible outcomes.
- IMIV.4.1.3 Distinguishes between independent and dependent compound events, and explains the idea of conditional probability.
- IMIV.4.1.4 Designs and uses trees, tables, area models, and other representational methods to calculate the probability of compound events in two-stage experiments when the events are independent and when they are not independent.
- IMIV.4.1.5 Describes and applies the multiplication rule for probability for computing probabilities for independent and for dependent compound events.

Objective IMIV.4.2:

Student develops, uses, and interprets simulations to estimate probabilities for events where theoretical values are difficult or impossible to compute.

Performance Expectations:

- IMIV.4.2.1 Describes a simulation by identifying the components and assumptions in a problem, selecting a device to generate chance outcomes, defining a trial, and specifying the number of trials; conducts a simulation.

IMIV.4.2.2 Summarizes data from a simulation using appropriate graphical and numerical summaries, develops an estimate for the probability of an event associated with a real-world probabilistic situation, and discusses the effect of the number of trials on the estimated probability of the event.

IMIV.4.2.3 Recognizes that simulation results are likely to differ from one run of the simulation to the next; observes that the results of the simulation tend to converge as the number of trials increases.

Standard IMIV.5: Bivariate Data and Trend-Line Models

Students examine patterns in scatterplots, and they develop models for trends in bivariate data using median-fit lines.

Objective IMIV.5.1:

Student analyzes bivariate numerical data, representing such data with appropriate scatterplots and sketched trend lines.

Performance Expectations:

IMIV.5.1.1 Judges whether a scatterplot appears to show a linear trend, and if it does, draws a trend line and writes an equation for that line; uses the equation to make predictions; and interprets the slope of the line in context.

IMIV.5.1.2 Computes the median-fit line, by hand, to model a relationship shown in a scatterplot, and interprets the slope and intercept in terms of the original context.

Integrated Mathematics V

STANDARDS

1. Polynomial Expressions, Functions, and Equations [Algebra]
2. Exponential, Logarithmic, and Other Functions [Algebra]
3. Structure of Sequences and Recursion [Algebra]
4. Trigonometric Functions [Geometry and Measurement]
5. Experiments, Surveys, and Observational Studies [Data Analysis]

Course goal:

Students explore polynomial, exponential, logarithmic, and trigonometric functions, and they apply these functions to investigate and model a variety of mathematical and real-world problems. Students develop recursive relationships and use them to model situations involving change in discrete settings. Students design and conduct experiments, surveys, and observational studies and communicate their findings.

Integrated Mathematics V is designed to develop students' knowledge of algebra and functions in three major ways. The first is to develop knowledge of polynomial equations and functions—their connections, their behavior, and their applications. The second is to examine nonlinear equations and functions—absolute value, rational, radical, logarithmic, and exponential functions—and their properties and applications. The third is to begin the study of trigonometric functions. These are developed based on the study of indirect measurement and the development and use of the Pythagorean theorem. Here, students are introduced to the basic trigonometric relationships along with their use in describing measures in right-triangle settings in other applications.

Students explore nonlinear functions, learning ways to investigate and describe their behavior. Students develop arithmetic and geometric sequences and related series, using a more-sophisticated approach than employed in Integrated Mathematics III. They model such sequences and series using recursion relationships and apply such recursive models to investigate and predict the long-term behavior of sequential patterns in a variety of applied settings. These applications involve data analysis and algebraic thinking and, in many cases, draw on knowledge from geometry and measurement.

Students' knowledge of statistical applications is extended by considering the design and analysis of experimental studies and by comparing and contrasting of surveys, experiments, and observational studies. Students use random allocation of experimental units to treatments in designing experiments, and they compare this procedure with random selection of units from a population for surveys.

Considerable emphasis is given to linking the work in this portion of the course with applications found in other areas of the secondary school curriculum and in real-world settings outside school.

The following integrated mathematics standards and objectives do not represent any specific curricular design for an effective integrated mathematics instructional program. This integrated framework is provided to support curriculum and instructional designers as they connect and integrate this content into effective instruction.

Standard IMV.1: Polynomial Expressions, Functions, and Equations

Students extend their understanding of functions from linear* settings to include polynomial functions, operations on these functions, and the solution of polynomial equations using complex numbers. Students use polynomials, especially quadratics, to model situations with graphical and symbolic representations, and they translate among these representations to represent and discuss the qualitative behavior of the associated functions.

Objective IMV.1.1:

Student operates with monomials, binomials, and polynomials, applies these operations to analyze the graphical behavior of polynomial functions, and applies the composition of functions to model and solve problems.

Performance Expectations:

IMV.1.1.1 Adds, subtracts, and multiplies polynomial expressions to solve problems.

IMV.1.1.2 Analyzes and describes graphs of polynomial functions by examining their intercepts, zeros, domain and range, and local (turning points) and global (end) behavior.

IMV.1.1.3 Uses factoring, properties of exponents, and knowledge of the related contextual needs to transform expressions and solve problems.

IMV.1.1.4 Applies the composition of functions to model and solve problems, explaining the results.

Objective IMV.1.2:

Student represents, compares, translates among representations, and graphically, symbolically, and tabularly represents, interprets, and solves problems involving quadratic functions.

Performance Expectations:

IMV.1.2.1 Identifies, interprets, and translates among different representations of quadratic functions, realizing that their graphs are parabolas.

IMV.1.2.2 Determines reasonable domain and range values for quadratic functions within a context, and tests the reasonableness of solutions to quadratic equations (zeros of quadratic functions).

IMV.1.2.3 Identifies any points of intersection of the graph of a quadratic equation of the form $y = ax^2$ and the graph of a line of the form $y = k$, and relates the points of intersection to the solutions of the quadratic equation $ax^2 = k$.

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*Underlined words and phrases are defined in the Glossary.

IMV.1.2.4 Sketches a quadratic function's graph, and recognizes the relationships between the coefficients of a quadratic function and characteristics of its graph (e.g., shape, position, intercepts, zeros, maximum, minimum, symmetry, vertex).

IMV.1.2.5 Formulates equations and inequalities based on quadratic functions, solves them using factoring, completing the square, and technology, and interprets their solutions in terms of the original problem context.

IMV.1.2.6 Develops the quadratic formula, and applies it to the solution of quadratic equations and the interpretation of the nature of their roots.

IMV.1.2.7 Constructs and solves quadratic inequalities in one and two variables, and represents their solutions graphically.

Objective IMV.1.3:

Student represents, applies, and discusses the properties of complex numbers.

Performance Expectations:

IMV.1.3.1 Defines, plots, and computes with complex numbers.

IMV.1.3.2 Describes how the associative, commutative, and distributive properties of operations on real numbers extend to operations on complex numbers.

IMV.1.3.3 Solves quadratic equations with real coefficients over the set of complex numbers.

Standard IMV.2: Exponential, Logarithmic, and Other Functions

Students develop exponential, logarithmic, and other nonlinear functions (rational, radical, absolute value, and piecewise-defined) to represent, investigate, and solve problems in mathematics and real-world contexts.

Objective IMV.2.1:

Student represents geometric or exponential growth with exponential functions and equations, and applies such functions and equations to solve problems in mathematics and real-world contexts.

Performance Expectations:

IMV.2.1.1 Extends the properties of rational exponents to real exponents, relating expressions with rational exponents to the corresponding radical expressions.

IMV.2.1.2 Approximates solutions to an exponential equation, and relates the solutions to the points of intersection of the graph of the exponential equation and the graph of a horizontal line.

IMV.2.1.3 Analyzes a problem situation modeled by an exponential function, formulates an equation or inequality, and solves the problem.

IMV.2.1.4 Uses exponential functions to solve problems involving compound interest and exponential growth and decay in mathematics and real-world contexts.

IMV.2.1.5 Graphs and analyzes the behavior of exponential functions.

Objective IMV.2.2:

Student defines logarithmic functions and uses them to solve problems in mathematics and real-world contexts.

Performance Expectations:

IMV.2.2.1 Defines a logarithm as a solution to an exponential equation, and recognizes the inverse relationship between functions defined by logarithms and exponential expressions, showing this relationship graphically.

IMV.2.2.2 Solves problems by applying properties of logarithms [$\log xy = \log x + \log y$; $\log\left(\frac{x}{y}\right) = \log x - \log y$, and $\log (x^a) = a\log(x)$] to construct equivalent forms of a logarithmic expression.

IMV.2.2.3 Applies the inverse relationship between exponential and logarithmic functions to solve problems in mathematics and real-world contexts.

Objective IMV.2.3:

Student interprets and represents rational and radical functions and solves rational and radical equations.

Performance Expectations:

IMV.2.3.1 Models and solves problems using <u>direct</u>, <u>inverse</u>, <u>joint</u>, and <u>combined variation</u>.

IMV.2.3.2 Models problem situations by constructing equations and inequalities based on rational functions, uses a variety of methods to solve them, and interprets the solutions in terms of the problem situation.

IMV.2.3.3 Adds, subtracts, multiplies, and evaluates rational functions and simplifies rational expressions with linear and quadratic denominators.

IMV.2.3.4 Describes the graphs of rational functions, describes limitations on the domains and ranges, and examines asymptotic behavior.

IMV.2.3.5 Uses properties of radicals to solve equations and identifies extraneous roots when they occur.

Objective IMV.2.4:

Student interprets and models step and other piecewise-defined functions, including functions involving absolute value.

Performance Expectations:

IMV.2.4.1 Analyzes a problem situation to determine or interpret reasonable domain and range values for piecewise-defined functions representing the situation.

IMV.2.4.2 Interprets, constructs, and applies step functions (e.g., greatest integer/floor) and other piecewise-defined functions, including absolute value functions, to model and solve problems.

IMV.2.4.3 Translates among verbal, graphical, tabular, and symbolic representations of step functions and other piecewise-defined functions, including absolute value functions.

Standard IMV.3: Structure of Sequences and Recursion

Students analyze and represent sequences and series and investigate how recursive relationships and their associated sequences can model the long-term behavior of situations involving sequential change.

Objective IMV.3.1:

Student categorizes sequences as arithmetic, geometric, or neither and develops formulas for the general terms and sums related to arithmetic and geometric sequences.

Performance Expectations:

IMV.3.1.1 Investigates the rate of change found in sequences, and uses it to characterize sequences as arithmetic, geometric, or neither.

IMV.3.1.2 Develops the general term for arithmetic and geometric sequences, and develops methods for calculating sums of terms for finite arithmetic and geometric sequences and the sum of a convergent infinite geometric series.

Objective IMV.3.2:

Student develops recursive relationships for modeling and investigating patterns in the long-term behavior of their associated sequences.

Performance Expectations:

IMV.3.2.1	Develops recursive relationships for arithmetic and for geometric growth situations.
IMV.3.2.2	Generates or constructs sequences from given recursive relationships modeling patterns found in mathematics and in other disciplines.
IMV.3.2.3	Investigates the long-term behavior of a recursive relationship, with and without technology.

Standard IMV.4: Trigonometric Ratios

Students develop and apply the Pythagorean theorem, right-triangle trigonometric ratios, and proportionality relationships in structuring and solving indirect measurement problems.

Objective IMV.4.1:

Student proves and applies the Pythagorean theorem and its converse, and student develops and applies the distance formula, properties of special right triangles, properties of proportions, and the basic trigonometric ratios.

Performance Expectations:

IMV.4.1.1	Proves the Pythagorean theorem and its converse and applies them in two- and three-dimensional settings.
IMV.4.1.2	Develops and applies the distance formula to determine the distance between points in the coordinate plane.
IMV.4.1.3	Develops and applies the properties of 30°-60°-90° and 45°-45°-90° triangles; develops and applies proportional relationships involving the altitude drawn to the hypotenuse of a right triangle.
IMV.4.1.4	Applies the sine, cosine, and tangent trigonometric ratios to determine lengths and angle measures in right triangles.
IMV.4.1.5	Applies, singly and in combination, the Pythagorean theorem, properties of proportionality, trigonometric ratios, and similarity in solving mathematical and real-world problems.

Standard IMV.5: Experiments, Surveys, and Observational Studies

Students distinguish among experiments, surveys, and observational studies. They design studies, collect and analyze data using appropriate methods, draw conclusions, and communicate results. They evaluate studies reported in the media.

Objective IMV.5.1:

Student identifies problems that can be addressed through collection and analysis of experimental data, designs and implements simple comparative experiments, and draws appropriate conclusions from the collected data.

Performance Expectations:

IMV.5.1.1 Describes how well-designed experiments use random assignment to balance the variation of some factors in order to isolate the effects of a treatment.

IMV.5.1.2 Designs a simple comparative experiment to answer a question: determines treatments, identifies methods of measuring variables, randomly assigns units to treatments, and collects data, distinguishing between explanatory and response variables.

IMV.5.1.3 Organizes and displays data from an experiment; summarizes the data using measures of center and spread, including the mean and standard deviation; identifies patterns and trends in tables and plots; and communicates methods used and the results of the experimental study to nontechnical persons.

Objective IMV.5.2:

Student distinguishes among surveys, observational studies, and designed experiments and relates each type of investigation to the research questions it is best suited to address. Student recognizes that an observed association between a response and an explanatory variable does not necessarily imply that the two variables are causally linked. Student recognizes the importance of random selection in surveys and random assignment in experimental studies. Student communicates the purposes, methods, and results of a statistical study, and evaluates studies reported in the media.

Performance Expectations:

IMV.5.2.1 Distinguishes among questions best explored through a sample survey, an observational study, or a designed experiment. Recognizes that an observed association between a response and an explanatory variable does not necessarily imply that the two variables are causally linked. Illustrates the different types of conclusions that may be drawn from surveys, observational studies, and designed experiments.

IMV.5.2.2 Evaluates possible factors involved in a given problem and what information they provide relative to the question of interest. Formulates specific questions and identifies quantitative measures that may be used in providing answers to the question of interest.

IMV.5.2.3 Describes advantages and disadvantages of using different methods of measuring variables. Explains how biases can occur in studies and their effects on study outcomes.

IMV.5.2.4 Compares and contrasts the random sampling of units from a population and the random assignment of treatments to experimental units.

IMV.5.2.5 Explains why most research questions do not have unique answers and why several approaches to answering the same question may be appropriate; explains why different studies of the same research question, conducted differently, may yield very different results and why this is to be expected.

IMV.5.2.6 Communicates, both orally and in writing, the purposes, methods, and results of a statistical study using nontechnical language.

IMV.5.2.7 Evaluates study results reported in the media.

Integrated Mathematics VI

STANDARDS

1. Properties of Families of Functions [Algebra]
2. Trigonometric Functions [Geometry and Measurement]
3. Conic Sections and Polar Equations [Geometry and Algebra]
4. Vectors and Parametric Equations [Algebra]
5. Least-Squares Regression and Association in Bivariate Data [Data Analysis]
6. Permutations, Combinations, and Probability Distributions [Data Analysis]

Course Goal:

Students develop, describe, investigate, and apply general function properties and use them to explore families of functions and parametric equations. Students develop the general trigonometric functions and analyze their properties while applying them in a wide variety of contexts. Students examine conic sections, polar representations, and vectors and apply them to a variety of mathematical and real-world contexts. They model bivariate data with regression lines and measure the linear association between two variables with the correlation coefficient. Students employ counting techniques to investigate, model, and solve problems.

In Integrated Mathematics VI, students are engaged in applying their study of mathematics to the modeling of mathematical and real-world situations. The processes of representation, connection, and communication are important factors in molding students' reasoning and problem-solving strategies.

Students extend their concept of function to operations with functions, including composition and translations, and an in-depth study of the qualitative behavior of functions from both a numerical and a graphical standpoint. They use a variety of functions—linear, quadratic, general polynomial, exponential, logarithmic, or trigonometric—appropriately to model change in different settings and to describe the qualitative behavior of various relationships. Students use their knowledge of functions to extend their data analytic abilities in developing trend predictions and assessing the appropriateness of such predictions in representing association in bivariate sets of data. The situations that are modeled may be derived from contexts based in number, geometry and measurement, or some scientific or business setting.

Trigonometric functions are extended from the study of right triangles to encompass side and angle measurements in all triangles through the use of the Law of Sines and the Law of Cosines. Students make the transition in this course from considering trigonometric functions based on degree measure to working almost exclusively in radian measure.

From this point forward, degrees are only used in situations involving specific degree-measure applications, such as solving particular triangles based on degree measure, as in the application of the Law of Cosines or problems involving angles of inclination or depression. The study of trigonometry is further extended to consider the circular functions associated with the angles centered at the origin of the unit circle. Students see the connections between the right-triangle, trigonometric-function, and unit-circle approaches and determine inverse functions for the primary domains of the circular functions.

The modeling skills developed in these areas are applied in the development of the general forms for the conic sections and of polar-coordinate representations for curves. The conics and polar representations are applied to modeling a number of mathematical and real-world relationships.

Vectors and parametric equations are developed to assist students in describing and representing the motion of objects in a plane. Applications of these concepts are used to formalize transformations and visually and symbolically model problems involving motion.

Students develop least-square regression lines to model trends in scatterplots of bivariate data. Students examine the correspondence between scatterplots of bivariate data and a related correlation coefficient. They also examine the leverage effects of individual data points that are not consistent with the overall trend in a scatterplot.

Students use permutations and combinations in counting. Students relate the distribution of outcomes from data and from probability experiments to the binomial and normal distributions.

The mathematics and statistics studied in this course provide a foundation for a wide variety of career paths that students may choose. This flexibility is important because students are moving forward to potentially a wide variety of coursework, ranging from the study of calculus, introductory statistics, or discrete/finite mathematics to applications of mathematics in various other disciplines or vocations. While these future paths each have specific prerequisites, the content for this course has been selected so as to provide students with a solid foundation for further study.

The following integrated mathematics standards and objectives do not represent any specific curricular design for an effective integrated mathematics instructional program. This integrated framework is provided to support curriculum and instructional designers as they connect and integrate this content into effective instruction.

Standard IMVI.1: Properties of Families of Functions

Students develop and apply properties of functions and families of functions and their related equations. Students apply and interpret the results of various operations with functions in mathematical and real-world situations.

Objective IMVI.1.1:

Student investigates behavior of functions and their related equations, and student compares and contrasts properties of families of functions and their related equations.

Performance Expectations:

IMVI.1.1.1 Determines the domain and range of functions as represented by symbols and graphs, where appropriate.

IMVI.1.1.2 Identifies and applies relationships among significant points of a function (zeros, maximum points, minimum points), the graph of the function, the nature and number of the function's zeros, and the symbolic representation of the function.

IMVI.1.1.3 Determines the number and nature of solutions to polynomial equations with real coefficients over the complex numbers.

IMVI.1.1.4 Recognizes and describes continuity, end behavior, asymptotes, symmetry (odd and even functions), and limits, and connects these concepts to graphs of functions.

IMVI.1.1.5 Identifies situations involving functions for which there is no elementary algorithm to find zeros (for example, $a^x = x^n$), and distinguishes them as such.

IMVI.1.1.6 Compares and contrasts characteristics of different families of functions, such as polynomial, rational, radical, power, exponential, logarithmic, trigonometric, and piecewise-defined functions, and translates among verbal, tabular, graphical, and symbolic representations of functions.

IMVI.1.1.7 Describes and contrasts common elementary functions symbolically and graphically, including x^n, x^{-1}, $\ln x$, $\log_a x$, e^x, a^x, and the basic trigonometric functions.

Objective IMVI.1.2:

Student examines and applies basic transformations of functions and investigates the composition of two functions in mathematical and real-world situations.

Performance Expectations:

IMVI.1.2.1 Finds, interprets, and graphs the sum, difference, product, and quotient (where it exists) of two functions, indicates the relevant domain and range for the resulting function, and provides a graph of the resulting function.

IMVI.1.2.2 Forms the composition of two functions, and determines the domain, range, and graph of the composite function. Composes two functions to determine whether they are inverses.

IMVI.1.2.3 Applies basic function transformations to a parent function $f(x)$, including $a \cdot f(x)$, $f(x) + d$, $f(x - c)$, $f(b \cdot x)$, $|f(x)|$, and $f(|x|)$, and interprets the results of these transformations verbally, graphically, and numerically.

Standard IMVI.2: Trigonometric Functions

Students extend trigonometric ratios to functions of angle measure and of real numbers. They develop these functions' graphs, properties, and inverse functions. Students solve trigonometric equations. They develop more-general trigonometric functions and apply them to solve real-world problems.

Objective IMVI.2.1:

Student solves problems involving measures in triangles by applying trigonometric functions of the degree or radian measure of a general angle and shifts from primarily viewing trigonometric functions as based on degree measure to viewing them as functions based on radian measure, and ultimately to viewing them as general periodic functions of real numbers. Student investigates the properties of trigonometric functions, their inverse functions, and their graphical representations.

Performance Expectations:

IMVI.2.1.1 Develops and applies the definition of the sine and cosine functions of the degree measure of a general angle in standard position* in relation to the values of the y- and x-coordinates, respectively, of points on the terminal side of the angle.

*Underlined words and phrases are defined in the Glossary.

IMVI.2.1.2 Develops radian measure of angles, measures angles in both degrees and radians, and converts between these measures.

IMVI.2.1.3 Defines the trigonometric functions as functions of the radian measure of a general angle, and describes them as functions of real numbers.

IMVI.2.1.4 Develops and applies the values of the trigonometric functions at 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ radians and their multiples.

IMVI.2.1.5 Constructs the graphs of the trigonometric functions, and describes their behavior, including periodicity, amplitude, zeros, and symmetries.

IMVI.2.1.6 Defines and graphs inverses of trigonometric functions with appropriately restricted domains.

IMVI.2.1.7 Develops the fundamental Pythagorean trigonometric identities, sum and difference identities, double-angle identities, and the secant, cosecant, and cotangent functions, and uses them to simplify trigonometric expressions.

IMVI.2.1.8 Develops the Law of Sines and the Law of Cosines, and uses them to find the measures of unknown sides and angles in triangles.

Objective IMVI.2.2:

Student uses transformations of trigonometric functions, their properties, and their graphs to model and solve trigonometric equations and a variety of problems.

Performance Expectations:

IMVI.2.2.1 Graphs functions of the form $f(t) = A \sin(Bt + C) + D$ or $g(t) = A \cos(Bt + C) + D$, and interprets A, B, C, and D in terms of amplitude, frequency, period, and vertical and phase shift.

IMVI.2.2.2 Relates and uses rectangular and polar representations of complex numbers, and uses DeMoivre's theorem.

IMVI.2.2.3 Solves trigonometric equations, noting the periodic nature of solutions when applicable, and interprets the solutions graphically.

IMVI.2.2.4 Uses trigonometric functions to model and solve problems in mathematics and other disciplines.

Standard IMVI.3: Conic Sections and Polar Equations

Students develop the general symbolic forms for and graphically represent the conic sections based on their locus descriptions, applying these results to mathematical and real-world contexts. Students represent points in polar form and find equivalent polar and rectangular representations for points and curves.

Objective IMVI.3.1:

Student develops and represents conic sections from their locus descriptions, illustrating the major features and graphs. Student uses conic sections and their properties to model mathematical and real-world problem situations.

Performance Expectations:

IMVI.3.1.1 Determines an equation representing each of the conic sections from its locus description.

IMVI.3.1.2 Analyzes a quadratic equation in x and y representing a conic with center at (h, k) and involving no rotation, recognizes the type of conic section represented, expresses the equation in a form useful for graphing, and constructs a graph of the conic.

IMVI.3.1.3 Uses conic sections to model and solve problems from mathematics and other disciplines.

Objective IMVI.3.2:

Student represents points and curves in rectangular and polar forms and finds equivalent polar and rectangular representations for points and curves.

Performance Expectations:

IMVI.3.2.1 Expresses points in the plane in both rectangular and polar forms.

IMVI.3.2.2 Finds equivalent representations for points and curves, including the conics, in both rectangular and polar forms.

Standard IMVI.4: Vectors and Parametric Equations

Students represent, investigate, and solve problems using two-dimensional vectors and parametric equations.

Objective IMVI.4.1:

Student applies vector concepts in two dimensions to represent, interpret, and solve problems.

Performance Expectations:

IMVI.4.1.1 Defines vectors in two dimensions as objects having magnitude and direction, and represents them geometrically.

IMVI.4.1.2 Illustrates and applies the properties of vector addition and scalar multiplication to represent, investigate, and solve problems.

IMVI.4.1.3 Uses vectors in modeling physical situations to solve problems.

IMVI.4.1.4 Models geometric translations with vector addition to solve problems.

Objective IMVI.4.2:

Student applies parametric methods to represent and interpret motion of objects in the plane.

Performance Expectations:

IMVI.4.2.1 Uses parametric equations to represent situations involving motion in the plane, including motion on a line, motion of a projectile, and motion of objects in orbits.

IMVI.4.2.2 Converts between a pair of parametric equations and an equation in x and y to interpret the situation represented.

IMVI.4.2.3 Analyzes planar curves, including those given in parametric form.

Standard IMVI.5: Least-Squares Regression and Association in Bivariate Data

Students develop models for trends in <u>bivariate data</u> using <u>least-squares regression lines</u>. Students use the correlation coefficient to measure <u>linear</u> association in scatterplots, and they examine the effects of outliers on the correlation and on models for trend. Students investigate the effects of data transformations on measures of center, spread, association, and trend.

Objective IMVI.5.1:

Student assesses association in tables and scatterplots of bivariate <u>numerical data</u> and uses the correlation coefficient to measure linear association. Student develops models for trends in bivariate data using least-squares regression lines.

Performance Expectations:

IMVI.5.1.1 Generates the least-squares regression line, using technology, to model a relationship shown in a scatterplot, and interprets the slope and intercept in terms of the original context.

IMVI.5.1.2 Determines the correlation, using technology, between two numerical variables, interprets the correlation, and describes the strengths and weaknesses of the correlation coefficient as a measure of linear association.

IMVI.5.1.3 Computes and plots residuals from the least-squares regression line; assesses the fit of the linear model using graphical and numerical results, and determines whether the use of a linear model is appropriate.

IMVI.5.1.4 Interpolates using trends observed in scatterplots or fitted regression lines, and judges when extrapolating observed trends may be appropriate.

Objective IMVI.5.2:

Student examines the influence of outliers on the correlation and on models for trend.

Performance Expectations:

IMVI.5.2.1 Identifies unusual observations in scatterplots, and conjectures about the effect of such outliers on the strength of the association between the variables defining the scatterplot.

IMVI.5.2.2 Investigates and describes the influence outliers may have on a correlation coefficient, on the slope and intercept of a least-squares regression line, and on a median-fit line.

IMVI.5.2.3 Analyzes the potential importance of outliers as flags for possible errors in the data, or as counterexamples or unique cases, especially when describing societal trends or behavioral traits.

Objective IMVI.5.3:

Student examines the effects of transformations on measures of center, spread, association, and trend and develops basic techniques for modeling and more-advanced data analytic techniques.

Performance Expectations:

IMVI.5.3.1 Demonstrates and describes how different scales (e.g., original, linear, square root, logarithmic) may affect scatterplots and summary statistics, and shows how different representations (tables, graphs, summary numbers) may reveal different characteristics of a data set.

IMVI.5.3.2 Describes and illustrates how data scales are chosen for convenience in analyzing and presenting information, and describes how transformations may be used in the development of linear models.

Standard IMVI.6: Permutations, Combinations, and Probability Distributions

Students solve ordering, counting, and related probability problems. Students recognize a binomial probability setting and compute the probability distribution for a binomial count. Students recognize settings where the normal model may be used, and they apply the empirical rule to solve problems.

Objective IMVI.6.1:

Student solves ordering, counting, and related probability problems. Student recognizes a binomial probability setting and computes the probability distribution for a binomial count.

Performance Expectations:

IMVI.6.1.1 Uses permutations, combinations, and the multiplication rule for counting (Fundamental Property of Counting) to solve counting and probability problems.

IMVI.6.1.2 Recognizes a binomial probability setting, and develops and graphs the probability distribution for a binomial count.

Objective IMVI.6.2:

Student identifies settings in which the normal distribution may be useful. Student describes characteristics of the normal distribution and uses the empirical rule to solve problems.

Performance Expectations:

IMVI.6.2.1 Identifies settings in which the normal distribution may be useful, and describes characteristics of the normal distribution.

IMVI.6.2.2 Uses graphical displays and the empirical rule to evaluate the appropriateness of the normal model for a given set of data. Uses the empirical rule to estimate the probability that an event will occur in a specific interval that can be described in terms of whole numbers of standard deviations about the mean.

Glossary

addition rule for probability Given two events A and B related to the same experiment, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

angle in standard position Angle positioned so that its vertex is at the origin, one side (which is considered the initial side) is the positive x-axis, and the other side is the terminal side. The quantity denoted by the rotation starting at the initial side and moving in the prescribed direction to the terminal side represents the measure of the angle or rotation associated with the general angle.

back-to-back stem-and-leaf plot A stem-and-leaf plot that displays the distributions of two variables simultaneously. The plot has a single stem. The leaves of one distribution are recorded on the right side of the stem and the leaves for the other distribution are recorded on the left side of the stem. Back-to-back stem-and-leaf plots can be used to display bivariate data when one of the variables is a categorical variable that indicates which population the data comes from (population 1 or population 2) and the other variable provides a measurement on each individual observed in each population (for example, heights of boys (population 1) and heights of girls (population 2) in a particular classroom).

binomial count The number of "successes" in n independent binomial trials. The count can take any value in the set $\{0, 1, 2, \ldots, n\}$.

bivariate data A data set in which two variables are measured and recorded for each subject (experimental unit).

box plot A display of the distribution of a set of data that is constructed from a five-number summary of the data (the minimum value, the first quartile (Q_1), the median, the third quartile (Q_3), and the maximum value). The points are arranged either vertically or horizontally against an appropriate numerical scale. A rectangular box is used to represent the region from Q_1 to Q_3 with a segment connecting the longer sides of the box at the median value. Segments extend from the ends of the box at either end to reach out to the minimum and to the maximum values, respectively. The rectangular box represents the middle 50 percent of the data set, and its length the interquartile range.

categorical data See *categorical variable*.

categorical variable A variable whose values form a finite, unordered, and discrete set. Also called a nominal variable. For example, gender is a categorical variable with values male and female.

census A sample consisting of the entire population of interest; in applications, an attempt to measure every member of an entire population.

combined variation A relationship that involves a combination of quantities that vary in both direct and indirect ways. For example, y varies directly with the cube of x and inversely with w in $y = \frac{kx^3}{w}$.

direct proof A proof proceeding from given information to the conclusion by successive steps of deriving intermediate conclusions from preceding steps. Proofs using the contrapositive of a statement are direct proofs in that they directly prove the validity of the contrapositive, which is logically equivalent to the desired statement.

direct variation The relationship between x and y when there exists a constant k such that $y = kx$.

empirical rule The rule of thumb stating that approximately 68% of the data in a normal distribution is within one standard deviation of the mean, approximately 95% is within two standard deviations of the mean, and more than 99% is within three standard deviations of the mean.

factor An explanatory variable in an experiment; for example, the type of drug and the dose level given to a subject may be two factors in an experiment.

Fundamental Property of Counting See *multiplication rule for counting*.

Fundamental Theorem of Arithmetic Theorem stating that every positive integer greater than 1 is either prime or can be expressed as a product of primes that is unique apart from the order of the prime factors.

histogram A graph used to display the distribution of values in a set of measurement data. The horizontal axis is divided into equal-length intervals that range from the smallest value in the data to the largest. Bars are drawn above each interval with the height of the bar giving the number (or percent) of data points contained in the interval. The area of each vertical bar represents the proportion of the data in the interval on which the bar sits.

HL congruence Result stating that two right triangles are congruent if their hypotenuses are congruent and they also share a pair of congruent legs.

indirect proof A proof of a statement by assuming the negation of the desired conclusion along with the given information and showing that this leads to a logically contradictory statement. Thus, the desired conclusion must be valid.

interquartile range (IQR) The distance between the first (Q_1) and third (Q_3) quartiles: IQR = $Q_3 - Q_1$.

inverse variation The relationship between x and y when there exists a constant k such that $y = \frac{k}{x}$.

isometric dot paper Graph paper where the rows of dots, indicating intersections of the lines, are offset to form a grid of vertices of equilateral triangles.

joint variation The relationship among x, y, and z when there exists a constant k such that $y = kxz$, that is, when one variable varies directly with the product of the other two.

least-squares regression line A linear model for bivariate data that minimizes the sum of squared deviations between the data points and the line.

line plot A univariate graphical display of the observed values of a variable in a given data set. The line plot consists of a horizontal line with a numerical scale underneath that represents the range of observed values. Dots are stacked above the line, over the values on the scale that correspond to numerical values observed in the data; one dot is plotted for each observation in the data set. The plot displays each observed value in the data set and shows the shape and location of the distribution of the data. Also called a dot plot.

linear The use of the word "linear" within this document follows the secondary school mathematics convention of referring to something whose graph in the coordinate plane is a line. This equates to the notion of an "affine relationship" in higher mathematics. This usage is consistent with other sets of recommendations for the secondary curriculum, and it does include the more restrictive use for "linear" found at the collegiate level. Hence, references to "linear relationships" in this document refer to relationships that can be modeled by equations of the form $y = ax + b$ where a and b are real numbers.

local axiomatic system A restricted set of axioms dealing with a restricted topic in geometry. Some of the axioms may be theorems in a more complete treatment of geometry, but are taken as axioms for the sake of studying the particular restricted, or local, topic.

mean absolute deviation A measure of variability (spread) in a set of data that is computed by determining the sum over all data points of the distance of a data point from the mean of the data and dividing this total by the number of data points; the average distance from a data point to the mean of the data.

measurement error Under- or overestimation of a population attribute that can be ascribed to the interviewer, the respondent, the survey instrument, or the method used for recording the data (e.g., through the use of poorly worded questionnaires or from missing or incorrect data).

median-fit line A linear model for bivariate data produced by dividing the horizontal axis of a scatterplot of the data into three (or more) segments of roughly equal lengths, finding medians of the data in each segment, and fitting a line to the medians. This technique has not been standardized, and several methods are extant.

multiplication rule for counting The principle that if a process consists of a set of steps carried out sequentially and if the first step may be completed in m ways, the second step in n ways, and so forth to the final step, which can be completed in z ways, then there are $(m)(n)\cdots(z)$ ways to complete the whole process.

multiplication rule for probability Rule whose special case states that given two independent events A and B, the probability of the event (A and B) is $P(A \text{ and } B) = P(A)\,P(B)$; and whose general form states that given any two events A and B, it is true that $P(A \text{ and } B) = P(A)\,P(B|A)$ and $P(A \text{ and } B) = P(B)\,P(A|B)$.

multistep word problem A problem in which the solver must find several (i.e., two or more) intermediate pieces of information, which must then be used in generating the solution to the entire problem.

mutually exclusive events Two events related to the same experiment that share no outcomes in common.

net A diagram showing a two-dimensional pattern that can be folded up to form the sides, edges, and vertices of a three-dimensional figure.

nonrandom sampling methods Techniques of selecting samples that may introduce bias in a sample. These include convenience samples (uses readily available subjects), judgment samples (uses subjects the investigator selects from the population), and volunteer samples (uses subjects who volunteer to be in the sample).

normal distribution A symmetric, unimodal, bell-shaped curve used to model many real-world phenomena. Also called the Gaussian distribution.

number sense Students capabilities to relate number characteristics to mathematical and real-world settings. This includes the ability to check reasonableness and to select the type of number needed to answer a question and the degree of precision appropriate. It involves the sense of whether an answer is reasonable and solves a problem in an adequate and meaningful way.

numerical data See *numerical variable.*

numerical variable A variable whose values can be ordered or compared by ratios. For example, temperatures of different substances or weights of objects.

one-stage experiment An experiment in which the student observes the outcome of a single random phenomenon, such as tossing a coin or drawing a card from a deck.

parallel box plots A graph that displays two or more box plots on the same scale for the purpose of comparing distributions.

piecewise-linear Describes a function whose graph in the coordinate plane consists solely of a union of line segments, rays, or both.

population parameter A number describing a characteristic of a population, such as the population mean, the population standard deviation, or actual proportions of groups within the population. Sampling is usually performed to compute statistics for the purpose of making inferences about population parameters.

rate A ratio expressing a comparison between two quantities having different measures, such as grams/dollar, miles/gallon, or dollars/Euro.

recursive relationship A relationship defined by giving its value at some initial point (usually 0 or 1) and a method of computing its value at point $n + 1$ using some or all of the values for the points up to and including n.

sample statistic A number describing a characteristic of a sample, such as an observed proportion, sample mean, sample median, or sample standard deviation, that is calculated from a sample without knowledge of the true values for the population. Sample statistics are often used to estimate their corresponding population parameters.

sampling error Under- or overestimation of a population attribute that occurs because only a subgroup of the population is measured or because particular subgroups of a population are missed in a sample.

simple random sample A sample obtained in such a way that all samples of the same size have the same chance of being selected. This may be thought of as all members of a large population having approximately the same chance of being selected.

standard deviation A measure of variability (spread) in a set of data that is computed by determining the square root of the average squared distance of all data points from the mean of the data.

stem-and-leaf plot A plot of data values that lie between 0 and 99 consisting of a stem of consecutive digits that represents the tens places of the data values and leaves that represent the units value of each observation in the data set. The stem is drawn vertically on the graph. Each leaf is placed on the graph on the horizontal line next to its stem value. For data sets with values between 100 and 999, the stem values may represent the hundreds digit and the leaves the tens digit for each data value, and the units digit is ignored. Stem-and-leaf plots are primarily used to display the shape of the distribution for small data sets, and they are computed by hand. These plots have the advantage that the numerical values of the data are included in the display and every value in the data set is displayed.

stratified random sample A sample obtained by classifying the elements in the population into mutually exclusive strata and selecting a simple random sample from each stratum.

synthetic geometry Geometry developed as a deductive system based on a set of postulates having points and lines as undefined terms and making no use of numbers and numerical measures as found in coordinate geometry.

treatment In an experiment, the conditions assigned to the experimental subjects. For example, one group of subjects might receive a drug at one dose level and another group might receive the same drug at a different dose level; the drug and its dose level constitute the treatment for each group of subjects.

triangle- and angle-inequality theorems The geometric theorems relating the positions of angles and sides of triangles based on the order of their measures: the longest side of a triangle is opposite the angle of greatest measure, and the shortest side of a triangle is opposite the angle of least measure.

two-independent-stage experiment See *two-stage experiment.*

two-stage experiment An experiment in which the student observes the outcomes of two random phenomena, such as tossing two coins or tossing a coin and then drawing a marble from an urn. The two random phenomena can be independent, such as the outcomes on two coins that are tossed, or the phenomena can be dependent, such as tossing a coin to determine which of two urns a marble will be drawn from at the second stage of the experiment.

unit rate Rate in which one of the entries is a 1, such as $9.75/1 lb.

univariate data A data set in which one variable is measured and recorded for each subject (experimental unit).

zero-product property Property that if $xy = 0$ for complex numbers x and y, then $x = 0$ or $y = 0$.

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