The College Board
Mathematical Sciences Framework

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The College Board Mathematical Sciences Framework[1]

The College Board Mathematical Sciences Framework provides an organizing structure to guide work of the College Board in the mathematical sciences. This work now includes the design of tests of mathematical knowledge and developed abilities, development of curriculum standards and materials for students in middle school and high school, and creation of professional development materials for mathematical sciences teachers. The Framework is organized around four *big ideas*, offering answers to the following questions:

I. What are the mathematical sciences?

II. What do we know about learning in the mathematical sciences, and how is that learning supported by technology and other tools?

III. What knowledge of the mathematical sciences and their teaching do teachers use to support learning, and how can they acquire this knowledge?

IV. How can assessment document and support learning?

The exploration of these big ideas—characterization of teaching, learning, and assessing in the mathematical sciences—is predicated on the College Board's mission of connecting students to college success and ever striving for the standards of equity and excellence.

***Prologue*: Equity and Excellence**

First and foremost, College Board work in the mathematical sciences serves its mission to connect students to college success and opportunity. Because of the fundamental and growing importance of mathematical know-how to the welfare of all people, this Framework emphasizes that connections to the mathematical sciences are essential for *all* students' learning and that success and opportunity are framed in the context of *excellence;* all students should have equal access to excellence in education in the mathematical sciences. Hence the banners of equity and excellence guide this work.

I. The Mathematical Sciences

School mathematics occupies a unique and peculiar place in American culture. Most argue that it is an essential subject, and it holds a prominent place in the K–12 curriculum. Yet many wonder what role mathematics plays in their own lives. School mathematics often bears little resemblance to the disciplines that make up the mathematical sciences. These disciplines—mathematics, statistics, computer science, various other areas of applied mathematical and computational sciences—reach every corner of twenty-first-century life, from the sciences and engineering, to art and music, to economics and the social sciences, to the pervasive presence of technology. The work of the College Board will be rooted in a vision of the mathematical sciences that promotes better understanding of their evolving role in society, in the everyday lives of citizens as well as in workplaces and research laboratories.

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[1] This document was drafted and adopted in 2007 by the College Board's Mathematical Sciences Academic Advisory Committee and should be reconsidered periodically by that Committee for revision.

Mathematics and the Mathematical Sciences

The earliest evidence of recorded mathematics dates back nearly 10,000 years, and academic mathematics, similar to that of today, dates back 2,500 years to 540 BCE, when Pythagoras is credited with proving the first great theorem (Eves, 1980). Recorded knowledge about mathematics is vast; applications are numerous, unpredictable, and fundamental to human endeavors. The discipline is so extensive and basic that it does not yield easily to definition, having been described as creative art (Halmos, 1968), science of patterns (Steen, 1988), science of order (National Research Council, 1984), and often in dictionaries as science of quantity and space. Many others have described mathematics in terms of roles it plays such as that of unifying and synthesizing of scientific knowledge (National Research Council, 1984) and its unreasonable effectiveness in the natural sciences (Wigner, 1969), indicating the many unanticipated applications of mathematics originally developed for its intrinsic value. Over the centuries, mathematics has nourished the rise of many disciplines in science and engineering, and in the past century, mathematics has become the linchpin of several disciplines that are often described as the mathematical sciences, which includes mathematics, statistics, operations research, actuarial science, and other areas of applied mathematics. Often, computer science, with historical roots in mathematics and electrical engineering, is included within the mathematical sciences. Here, as an organization of disciplines within programs at the College Board, the mathematical sciences are mathematics, statistics, and computer science.

Today, mathematics is a fundamental and critical part of school and college, present in every grade of school and second only to English in college course enrollments. In recent years, the mathematics curriculum in K–12 has included statistics as one of its strands, and statistics and computer science have become staples in college offerings both as majors for degrees and as general education courses.

Education in the mathematical sciences serves myriad purposes. Knowledge of the mathematical sciences is essential in an increasingly quantitative world awash with data and analyses of data. Knowledge is needed for activities of citizens, consumers, and workers, as well as for developments in other academic disciplines including business, social sciences, sciences, and engineering. This, along with the need for educating future mathematical scientists to continue expanding the scope of mathematical knowledge, constitutes one of the major challenges in U.S. education—deciding what mathematics content to teach, how to teach it, and for what purposes.

Statistics

The discipline of statistics has a relatively short history as compared with the long history of the development of mathematics. Study of the properties of chance events began in mid-seventeenth century, when Blaise Pascal and Pierre de Fermat formulated the beginnings of probability, the basis of chance and risk (Bernstein, 1996, p. 3). Statistical analysis of data became widely used in the sciences, especially in astronomy, in the early nineteenth century. The characteristics that set statistics apart from mathematics are its emphasis on inductive reasoning, its focus on the variability in observed measurements, and its caveat that the context in which data are collected is essential to the interpretation of patterns found in the data. Statistical reasoning is primarily an inferential process; it focuses on reasoning about complex systems in settings in which the available information consists of measurements made with error. Statistical conclusions about a complex system are never known with certainty. New evidence, often in the form of new data, may change previously accepted conclusions.

The modern discipline of statistics organizes and governs empirical research. Stigler (1986) defines statistics as "a logic and methodology for the measurement of uncertainty and for an examination of the consequences of that uncertainty in the planning and interpretation of experimentation and observation." The study of statistics and probability has become increasingly important as more areas of the sciences and of the social sciences have adopted quantitative methods in the study of their disciplines. Knowledge of statistics and probability has become essential for fields as diverse as anthropology, astronomy, biology, chemistry, economics, engineering, geology, medicine, physics, political science, psychology, and sociology.

Because of the ubiquity of statistical analyses of data and their use in decision making in almost all areas of human activity, statistical literacy, like mathematical and computing literacy, is essential for everyone as consumers, citizens, and professionals. Thus, statistics education, like mathematics and computer education, has dual roles, being fundamental to scientific work and essential for understanding the contemporary world by all. Serving in these dual capacities constitutes a major challenge to mathematical sciences education and, more generally, to education across the disciplines. This larger challenge to education, often under the banner of quantitative reasoning or critical thinking, requires cooperation across all disciplines, and building that cooperation and the resulting educational synergy has to be foremost as College Board programs evolve.

Computer Science

The umbrella of mathematical sciences often extends to disciplines that have at their core mathematics or mathematical modes of thought. Computer science is one such discipline. Algorithms and problem solving are at the heart of the discipline of computing; the computer serves as a tool for implementing solutions.

Like statistics, computer science applies mathematical principles to investigate every area of human endeavor. Both statistics and computer science are interesting subjects in their own right, but the primacy of both in the mathematical sciences comes from their ability to use mathematical principles as tools for solving problems. Mathematics and mathematical modes of thought are the foundations on which both are built.

Computer science, however, makes different use of mathematics than does statistics. A look at AP® Statistics and AP Computer Science examinations illustrates this difference. The mathematical aspects of statistics are apparent from the questions on any AP Statistics examination, while the mathematical basis for the subject may not be as obvious from the questions on a typical AP Computer Science examination. Computer science makes extensive use of the internal structures of mathematical thought rather than the more familiar external structures of symbolic representations of algebraic expressions and equations. Problem formulation and problem revision are crucial when the ultimate goal is problem solving. Conceptual understanding of mathematical ideas and facility in mathematical thinking are essential for success in computer science.

The study of computer science requires a solid understanding of functions and function notation, of procedures, operations, and algorithms, of variables and parameters, of input-output and domains and ranges. Recursion and iteration are fundamental, as are matrix structures, trees, graphs, and countless other mathematical constructs and principles. Students cannot leave computer science without a rich understanding of logical organization and critical analysis. All of these we call mathematical modes of thought. Compared with mathematics, and even with statistics, computer science is a relatively young academic discipline, with its earliest roots in the theoretical work of Alan Turing, Alonzo Church, and others in the 1930s (Lewis & Papadimitriou, 1981). Digital computers came along in the 1950s and 1960s and became commonplace in higher education during 1970–1990, when academic programs in computer science developed in colleges and universities. Since then, computers have become a major part of most areas of human activity, and computer literacy a requirement for everyday life. Key computer science concepts such as algorithmic complexity permeate current real-world problems across disciplines. A fundamental understanding of computers and the discipline of computing is therefore an essential component of mathematical science education alongside mathematics and statistics.

Technology

Clearly, computers have always been fundamental in computer science, but they also have had major impacts on mathematics and statistics. Computers enabled the compiling and analysis of large data sets, radically transforming and expanding the role of statistics in education and other parts of society. Calculators, which are really small handheld computers, facilitated and promoted data analysis and statistical methods in K–12 education. Applications of mathematics and mathematics education, too, were significantly changed by computers and calculators. Even research in core mathematics was facilitated by the computation and visualization made possible by computers. Graphing calculators, some with computer algebra systems, have become commonplace in secondary school and college mathematics classrooms and on College Board assessments. Calculators and computers change how one teaches and assesses, and they change how students learn (see, for example, Fey, Cuoco, Kieran, McMullin, & Zbiek, 2003), so the role and effect of technology will be threaded throughout this framework.

II. Learning and Learners

How do people learn, and in particular, how do they learn mathematics? In the past thirty years, we have come to understand learning as a complex set of psychological and social processes by which individuals come to understand the experiences that constitute their world (Bransford, Brown, & Cocking, 2000). Our current understanding of learning includes the following features:

- Learning with understanding is supported when knowledge is structured around the major concepts and principles of the discipline.
- Learners use what they already know to construct new understandings.
- Learning is facilitated by use of metacognitive, or reflective, strategies that assist learners in identifying, monitoring, and regulating their cognitive processes.
- Learners have different strategies, approaches, patterns of developed abilities, and learning styles because of interactions among their opportunities to learn and their prior experiences.
- Learning is situated in activity and is shaped by the context and culture in which it occurs.
- Learning is enhanced through socially supported interactions.

- Learners' motivation to learn and their sense of self affect what they learn, how much they learn, and how much effort they will put into the learning.

The central goal of schooling in the mathematical sciences is the development of mathematical proficiency for all students, and effective mathematics instruction begins with an understanding of how students learn mathematics. Mathematical proficiency is now understood to be much more complex than simple mastery of skills. The National Research Council's *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001), although focusing on mathematical proficiency in grades pre-K to 8, outlines a sorting that seems to apply at all educational levels. Mastery of skills or procedural fluency—being able to carry out procedures flexibly, accurately, and efficiently, and knowing when such procedures are appropriate—is but one of several intertwined threads of proficiency described in *Adding It Up*. The complete list of the five threads of mathematical proficiency, with brief descriptions, is as follows:

- **Conceptual understanding—**comprehension of mathematical concepts, operations, and relations
- **Procedural fluency—**skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic competence—**ability to formulate, represent, and solve mathematical problems
- **Adaptive reasoning—**capacity for logical thought, reflection, explanation, and justification
- **Productive disposition—**habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

This mathematical proficiency is our goal for all students.

In the past thirty years, not only have we come to better understand how people learn, our understanding of how children come to know important mathematical ideas has grown tremendously. Learning mathematics is not simply the accrual of knowledge, acquired through listening and practicing. Rather, learning has both individual and social components, characterized by two key principles:

1. **Learners actively construct knowledge.** Learners make their own sense of new experiences through the lenses of their own prior experiences. This is an active process; new information either reinforces existing mental structures or challenges the learner to make new connections. Students' work makes sense to themselves. For example, the error $(x + y)^2 = x^2 + y^2$ is so common precisely because it makes sense to many students for whom this set of symbols looks like $2(x + y) = 2x + 2y$. Only experiences that challenge this surface understanding of the symbols are likely to challenge learners to restructure their knowledge of, in this case, exponents and the distributive property.
2. **Learning experiences are rooted in social and cultural context.** Learning occurs through social interactions among learners and teachers, engaged in mathematical activity in particular places at specific times. By participating in this activity, learners trade in the currency of ideas. They negotiate meaning, evaluate ideas' generality and usefulness, and assess arguments' persuasiveness. Through communicating solutions and strategies for solving problems, students in a class

> can work to convince each other (and themselves) of their approaches' correctness, uniqueness, and robustness. Individual students will construct their own individual understandings of mathematics based on participation in this conversation. They will also come to know what mathematics is as a discipline and what it means to do mathematics.

This individual and social understanding of the complex processes of learning mathematics has implications for the design of curriculum, the organization and conduct of classroom instruction, and the assessment of students' knowledge. It implies

- a curriculum that stresses all five threads of proficiency, that invokes a variety of representations and models, and that emphasizes connections among representations of important concepts and among concepts;
- instruction that engages students in exploration, pattern-seeking, explaining, extending and generalizing, justifying and proving; and
- an assessment scheme that is aligned with the curriculum and instruction described above, and that both documents and supports the development of all threads of mathematical proficiency.

III. Teaching and Teachers

The basis of children's engagement in mathematics in school is largely their classroom experiences and the actions of their teachers. Rich learning experiences are structured and orchestrated by teachers who know the discipline, who understand students and the nature of learning, and who possess a vast array of instructional strategies for fostering and monitoring learning. The College Board's programs for teachers will be informed by the following conceptions of what teachers must know to teach mathematics well, and how they build this knowledge.

Mathematics for Teaching

Mathematical knowledge for teaching includes a deep and clear understanding of the mathematics to be taught and broad knowledge of the mathematical sciences. Teachers need a thorough understanding of curriculum materials, including the key ideas that underlie these materials and how students come to understand specific mathematical ideas. In addition to this requisite knowledge, there are many talents, skills, and personal gifts that characterize a successful teacher. Although this has been clearly demonstrated to be false by years of research, the widespread idea prevails (even among many educators) that anyone who can "do" mathematical subjects like high school algebra can successfully teach them. The fact that such success is often measured with superficial assessments that require only short-term acquisition of manipulative skills has contributed to this unfortunate misunderstanding. When success is measured in terms of higher cognitive skills (understanding mathematics, solving unfamiliar problems, formulating conjectures and strategies), students learn better from teachers who can draw on a richer mathematical background. In particular, teachers need a broad, deep, and clear understanding of mathematics in order to accomplish the following:

- **Distinguish correct student work from incorrect student work.** When students are allowed some freedom in choosing how to solve a problem, they often proceed in unexpected ways. A teacher who is focused on a particular path to a solution might assess the validity of a student's method on the basis of the answer, running the risk of penalizing good mathematics or rewarding bad mathematics. Even worse, a teacher might try to avoid such a scenario by insisting that students solve problems the "right" way, dismissing all alternative approaches as simply unacceptable.

- **Encourage students to ask questions.** The natural curiosity of students can be a teacher's greatest asset, but only if the teacher can understand a student's question and use it to promote further learning. This often requires an understanding of mathematics that is not found in the textbook, perhaps not even in the course.
- **Continually provide a long-range mathematical perspective for the students.** It is difficult for students to see big ideas in mathematics when their learning is delivered in discrete daily lessons reinforced by focused homework assignments that do not often relate to each other. The teacher is the one who must pull things together for the students. Only a teacher who appreciates the interconnectedness of the mathematics throughout the curriculum can successfully provide the needed perspective.
- **Guide students to concerted implementation of all five strands of mathematical proficiency.** So much emphasis is placed on the "doing" of mathematics—procedural fluency—that the other four strands of proficiency can often be seen as irrelevant goals. While this might arguably be true in the short term (spawning the long heritage of ad hoc mechanical algorithms and mnemonics found in school mathematics today), conceptual understanding, adaptive reasoning, strategic competence, and productive disposition are all crucial for long-term success in the mathematical sciences. Thus, while some teachers might think they are being more efficient by showing students what to do and letting them follow their lead, it is the teacher who can lead learners to the more complete proficiency that in the end will better serve students.
- **Encourage creative thinking as part of the learning process.** Only a teacher who can dare to welcome the unexpected is likely to encourage students to think creatively in mathematics. Historically, it is teachers with a shallow understanding of their subjects who have created the metaphorical limiting "box" outside of which people are too seldom encouraged to think.
- **Teach mathematics with confidence.** A teacher with a strong mathematical background can teach mathematics with confidence, secure in the verifiable truth that the subject ideally provides. This confidence is communicated to the students, just as a successful coach communicates it to a team. In a confident classroom environment, students can explore, appreciate, and enjoy mathematics, secure in the knowledge that their correct work will be recognized and their natural curiosity satisfied.

Pedagogical Knowledge for Teaching Mathematics

Pedagogical knowledge is understanding how to represent and formulate content in ways that make that content comprehensible to others. Further, pedagogical knowledge includes using content knowledge to assess what students know about the content, how they know it, and how to use this information effectively to increase student understanding. Pedagogical knowledge for teaching mathematics informs teachers' design, enactment, and assessment of the day-to-day work of teaching. After years of virtually ignoring pedagogy as an unnecessary consideration in the teaching of mathematics, mathematics educators have recently taken the lead in reforming the American approach to teaching in order to better conform to how students actually learn. This new iconoclasm has led to considerable debate among the members of the mathematical community (many of whom represent the greatest successes of the former pedagogy), and in the tradition of academic discourse, this debate ultimately served to refine the best practices that have emerged. Many now believe that teacher-oriented practices like lecturing have been ineffective in reaching a majority of students, and that traditional assessment methods have failed all students (sometimes literally) in myriad ways. Today's mathematics teachers are adapting to the realization that, no matter how well the material is presented in class, student learning depends on other factors that the teacher can, and therefore should, control. In particular, a teacher should

- **Conduct a student-oriented classroom.** To the greatest extent possible, students should be doing more mathematics in the classroom than the teacher is. This is an unfamiliar approach for many classically educated teachers, but it can be learned. Even with large classes, grouping students together and giving them focused activities can create an active learning environment that is far more effective than the passive environment that prevailed a decade ago.
- **Engage students with probing questions that can inform instruction.** Sometimes called formative assessment, the practice of continually gathering feedback on what students are learning is one of the most powerful tools at a teacher's disposal. Effective formative assessment, some art and some science, can be learned and can be enhanced with practice.
- **Make students understand that they can do meaningful mathematics outside the classroom.** While this is obviously the optimal outcome of any mathematics course, it is sobering to consider how many traditional classroom practices have been counterproductive in that regard. Empowering students is different from informing students and, therefore, requires a different pedagogy.
- **Incorporate collaboration into the teaching and learning of mathematics.** Not only do most students learn more effectively when they collaborate with other learners, but teachers frequently teach more effectively when they collaborate with other teachers. Encouraging collaboration is a matter of pedagogy. By collaborating with their own colleagues to improve their teaching, teachers are modeling lifelong learning for their students.
- **Make judicious use of technology to enhance the teaching and learning of mathematics.** The use of technology in the mathematics classroom is now seen more as a matter of pedagogy than as a matter of course content. Technology empowers more students to do more mathematics and exposes them to the conditions under which they will eventually be expected to do it; nevertheless, the goal is still to teach mathematics effectively, not to teach the use of a particular technology.
- **Develop connections to other disciplines.** Learning mathematics can be encouraged and enhanced by observing and articulating connections to and uses in other

disciplines. Differences in terminology and conceptual formulations should be reconciled to whatever extent possible.

Professional Development

Professional development in the teaching of the mathematical sciences is the process of ongoing growth in content knowledge and pedagogical skill and in the relationship between content and pedagogy centered on increasing effectiveness in student learning of the mathematical sciences. The need for professional development for teachers is based on the realization that professional growth requires time and constant attention, and is an activity that extends over the teachers' careers. Professional development should be grounded in classroom practice and centered on student learning of the mathematical sciences. There are four critical and interconnected components involved in the process of teaching and learning in the mathematical sciences that should be the focus of all teacher professional development in the mathematical sciences. These components are content, pedagogy, assessment, and equity.

Content—Professional development programs should center on extending and deepening teacher content knowledge (Wu, 1999) to support effective organization of content instruction, assessment of student content understanding, and advancement of student content learning. Professional development directed at dealing with teachers' conceptions of content should be based on the recognition that knowing mathematics and knowing how to teach mathematics are two different things. Professional development must also be based on recognizing that teachers' content understanding must be global and more than simply knowing the content of the course or grade level that they teach. Teachers must understand how the content that they teach fits into the rest of school mathematics, what its long-term importance is, and what the value of this content is to the students. These require the teacher to understand central, unifying concepts and principles of the mathematical sciences (Ma, 1999).

Pedagogy—Student thinking in the mathematical sciences is the key to student learning. Professional development programs should center on strategies that promote and encourage student thinking, as well as on strategies that promote student communication to confirm what and how they are thinking; student communication thus provides a window into what students know and the depth of their understanding in the mathematical sciences. Professional development programs should also provide teachers with opportunities to take an active role in trying out various pedagogical strategies and approaches and to have a chance to recycle their attempts, supported by mentored feedback, to hone their practice with new methods of teaching and with improving their skill at assessing student thinking.

Assessment—The role of assessment in the practice of classroom teaching in the mathematical sciences is twofold: to monitor and confirm student acquisition of knowledge and proficiency; and to inform instructional strategies for deepening and extending student knowledge and proficiency (National Council of Teachers of Mathematics, 1995). Tied to these roles are two types of assessment: summative assessment and formative assessment. Confirming student understanding and proficiency is typically accomplished by summative assessments, assessments that quantitatively summarize student learning. Summative assessments tend to be "snapshots" of student learning and provide little or no useful feedback to teachers on next steps for improving student learning. In contrast, formative assessment is an integral part of the teaching and learning process. Formative assessment involves the continual monitoring of what each student knows through in-class communication of student thinking, presented orally or in writing, and uses this student feedback to inform what follow-up instructional strategies are needed to deepen and increase student learning (Black & Wiliam, 1998). Professional development programs should emphasize the value of formative assessment to the practice of teaching, strategies for obtaining valuable student feedback to aid the practice of formative assessment in the classroom, and the application of content knowledge to assessing what students know about content and how they know it, and to using this information effectively to increase student understanding.

Equity—Professional development programs should address equity and excellence, not as a statement or slogan that passively proclaims support for access to rigorous content and high expectations for all students, but by actively linking equity and excellence to the context of contemporary classroom practice, the mathematical sciences curriculum, and the school environment. In particular, equity and excellence must be a professional development strand that examines and advances methods and strategies in content, pedagogy, and assessment that teachers can apply directly to their classrooms and are geared to improving their effectiveness in increasing student learning in the mathematical sciences.

IV. Assessment

The College Board has been involved for decades in the design and administration of large-scale assessments such as the SAT Reasoning Test™ and tests administered as part of the Advanced Placement Program®. The purpose of these assessments has been to document knowledge or measure reasoning for the purposes of college admissions or credit in college courses. However, recent programs such as SpringBoard® have included assessments primarily designed to inform instruction and students, shifting the College Board assessment programs more toward formative assessment as opposed to summative assessments.

In school and college classrooms around the world, assessment goes on daily, formally and informally, as teachers interact with students individually and in groups, and assign problems, questions, quizzes, tests, homework, and projects, all in order to monitor student progress, provide feedback, and evaluate their own teaching. However, all assessment, whether formal or informal, large-scale or classroom, is grounded in the discipline, informed by how students learn, and designed to achieve the following goals:

- Assessment must support the learning of important mathematics, statistics, and computer science.
- Assessment must inform and guide instruction by giving teachers an opportunity to learn what students know and are able to do.
- Assessment must be aligned with curriculum and instruction.
- Assessment must focus on conceptual understanding as well as procedural skills.
- Assessment must give all students opportunities to show what they know and are able to do.

Technology and Assessment

The use of technology, mostly computers and calculators, has significantly affected assessment, especially large-scale testing. For example, the AP Calculus examinations changed considerably when graphing calculators were first required in 1995; many of these calculators also have computer algebra systems. Calculator support is essential for assessment in AP Statistics, and AP Computer Science would not exist as a course absent computer technology. The presence of technology introduces possible inequities because various calculators have different capabilities, and students with access to technology often have advantages over students without such access. Consequently, examination items must be carefully constructed so as to minimize the inequities. Other changes that technology prompted are different kinds of items, different values for parts of responses, questions about forms of answers, and new difficulties in evaluating student responses (Cannon & Madison, 2003). Teachers, test developers, and program administrators must ensure that all students have equal opportunities for access and knowledge about using technology in assessment, which requires equal opportunities for using technology in learning.

Epilogue: Challenge of Equity and Excellence

In the context of teaching and learning in the mathematical sciences, equity and excellence mean high learning expectations for all students, excellence in teaching, and the application of appropriate strategies for supporting high levels of learning for all students. To implement equity and excellence, teachers face a daunting task of organizing instruction for classrooms that typically contain a diversity of students. These diversities encompass issues related to students' ability to learn, such as socioeconomics, ethnicity, and race, English language learners, learning disabilities, varying levels of prerequisite knowledge and varying remediation needs, and even quality of life. Classrooms with such diversities require heightened attention to extending access to rigorous mathematics and setting high expectations for all students. Teachers need to know that there are strategies and practices that can help when encountering these diversities. Examples of such strategies are

- **Emphasize all five strands of mathematical proficiency.** According to research on how people learn, "experts' knowledge is not simply a list of facts and formulas … instead, their knowledge is organized around core concepts or 'big ideas'" (Bransford, Brown, & Cocking, 2000, p. 36). Conceptual understanding provides all students with deeper and longer-term understanding that supports and motivates their skill and procedural fluency, and their acquisition of mathematics learning. Strategic competence broadens students' use of mathematics, and adaptive reasoning not only deepens understanding but provides logical connections and explanations for mathematical work. Together, all this causes the disposition of students toward doing mathematics to grow.
- **Emphasize thinking as the key to learning.** A fundamental principle of learning is that students learn when they are guided to construct new understandings, linked to existing knowledge, and to reflect on their cognitive processes.
- **Emphasize communication in mathematics as the key to confirming student thinking in mathematics.** Thinking is key to students' learning, but communication confirms whether or not students are thinking, what they have thought about, and how

they have thought about it. Communication with and by a student, both oral and written, provides a window into what a student knows and the depth of a student's understanding.

- **Emphasize formative assessment.** Formative assessment is the strategy of obtaining information about what students know and how they know it within the daily practice of classroom instruction. Formative assessment begins with getting students to think about mathematics through the use of rich questions, problems, tasks, or performance-based activities, and it relies on getting students to communicate their thinking related to such classroom activities. Formative assessment practices provide teachers with direct and immediate information about what individual students know about mathematics and how they know it. Armed with valuable information of this sort, teachers can individualize instruction and use this information to raise student understanding to a higher level.
- **Emphasize multiple representations as a standard teaching practice.** The use of multiple representations (words, tables, symbols, graphs, manipulatives, models, technology, etc.) to explain, investigate, explore, analyze, interpret, and understand mathematics in a variety of connected representational forms offers an effective instructional strategy for reaching out to all students and their diverse learning styles.
- **Emphasize mathematics in context.** The utility, power, and beauty of mathematics arise from its ability, by virtue of its abstractness, to find connections across and between seemingly disparate concrete situations. Real-world and concrete contexts provide the foundation for discovering and forming the abstract patterns and models of mathematics. Students more effectively learn the power of mathematics by starting with their investigations of the "big ideas" of mathematics in real-world and concrete contexts.

Equity and excellence in mathematical sciences teaching recognizes that, within a diverse classroom setting, every student should expect a learning experience that bolsters their vision of themselves as confident and productive learners in the mathematical sciences.

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